Linear time algorithm for Quantum 2SAT

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Background
Classical 2SAT

A classical 2SAT instance $\Phi$ is a Boolean formula defined on
- a set of $n$ variables: $\{x_1, \ldots, x_n\}$
- as a conjunction of $m$ clauses: $\{C_1, \ldots, C_n\}$ where
- each clause is an OR of at most 2 literals (i.e. $x_i$ and $\overline{x_i}$)

Goal: Find an assignment to the variables so that $\Phi$ evaluates to true.

Example (2SAT)

An instance: $\Phi = (x_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2) \land (x_1 \lor \overline{x}_4) \land (x_4) \land (x_2 \lor x_3)$

Satisfying assignment: $a = 1011$
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Algorithms for 2SAT


Both algorithms have an optimal $O(n + m)$ running time.
Quantum 2SAT (2-QSAT)

A quantum 2SAT instance $\mathcal{H}$ is a 2-local Hamiltonian defined on

- $n$ qubits: $\{x_1, \ldots, x_n\}$
- as a sum of $m$ local terms: $\mathcal{H} = \sum_{uv} \Pi_{uv}$ where
- each $\Pi_{uv}$ is a projector acting non-trivially on qubits $(u, v)$;

Example (Q2SAT)

A 2-QSAT instance: $\mathcal{H} = \Pi_{12} + \Pi_{23} + \Pi_{34}$ with

$\Pi_{12} = |00\rangle\langle 00| + |11\rangle\langle 11|; \quad \Pi_{34} = |\Psi^-\rangle\langle \Psi^-|; \quad \Pi_{23} = |01\rangle\langle 01|;$
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- each $\Pi_{uv}$ is a projector acting non-trivially on qubits $(u, v)$;
- the smallest eigenvalue of $\mathcal{H}$ is its ground energy and
- the corresponding eigenvector $|\psi\rangle$ is the ground state of $\mathcal{H}$.

Goal: Given $\mathcal{H}$, output a ground state if the ground energy is 0 or "Unsatisfiable" otherwise.

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Prior Work
An $O(n^4)$ 2-QSAT algorithm by Bravyi (2006) based on finding the transitive closure of a directed graph.

- For $k \geq 3$, $k$-QSAT is $\text{QMA}_1$-complete
Solving 2-QSAT

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Our Result
An optimal $O(n + m)$ time algorithms for 2-QSAT (w.r.t # of operations over complex numbers).
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- Quantum variant of the EIS Algorithm
  - Infers a qubit assignment and propagates it throughout the system.
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Related Work
- Quantum analogue of the APT algorithm [dBG]
  - Uses the notion of transfer matrices to mirror the implications in Boolean Formulae.
Bravyi’s algorithm uses the following approach:

- For every qubit triple \((i, j, k)\), if there is a constraint on \((i, j)\) and \((j, k)\), add an implied constraint acting on \((i, k)\).

- Takes \(O(n^3)\) time to add all implied constraints.

- Requires globally affecting the instance and manipulating it.
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The linear time algorithms approach an instance by:

- Analyzing only a part of the instance and manipulating it with local operations.

- Local sections of the instance on being solved are decoupled from the rest of instance.

- Governed by graph traversals that can be executed in linear time.
Algorithm Building Blocks
Assume WLOG that $\mathcal{H}$ contains rank-1 and rank-3 projectors only.
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The constraint graph, $G(\mathcal{H})$ is defined as having
- $n$ vertices representing each qubit and
- labeled edges $\Pi_{ij}$ between $(i,j)$ for each term in $\mathcal{H}$

Example (Constraint Graph)

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**Theorem (Product State Theorem [CCD+11, ASSZ15])**

Any satisfiable 2-QSAT instance has a ground state which is a tensor product of \textbf{one qubit and two-qubit states}, where two-qubit states only appear in the support of rank-3 projectors.
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Ground State $= |\Psi^+\rangle_{12} \otimes |0\rangle_3 \otimes |1\rangle_4$

**Theorem (Product State Theorem [CCD$^+11$,ASSZ15])**

Any satisfiable 2-QSAT instance has a ground state which is a tensor product of one qubit and two-qubit states, where two-qubit states only appear in the support of rank-3 projectors.
Definition (Propagation)

Let $\Pi_{ij} = |\psi\rangle\langle\psi|$ be a rank-1 projector and $|\alpha\rangle$ be the state assigned to $i$. Then, $\Pi_{ij}$ propagates $|\alpha\rangle$ if, up to a phase, there exists a unique 1-qubit state $|\beta\rangle$ such that $\langle\psi|(|\alpha\rangle_i \otimes |\beta\rangle_j) = 0$. 

An entangled constraint always propagates every state. A product constraint will not propagate a state when already satisfied.
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\[ i \quad |\gamma\delta\rangle\langle\gamma\delta| \quad j \]

\[ |\psi_i\rangle \quad |\psi_j\rangle \]

\[ \neq |\gamma^\perp\rangle \quad \Rightarrow \quad = |\delta^\perp\rangle \]
**Propagating States**

Let $\Pi_{ij} = |\psi\rangle \langle \psi|$ be a rank-1 projector and $|\alpha\rangle$ be the state assigned to $i$. Then, $\Pi_{ij}$ propagates $|\alpha\rangle$ if, up to a phase, there exists a unique $1$-qubit state $|\beta\rangle$ such that $\langle \psi | (|\alpha\rangle_i \otimes |\beta\rangle_j) = 0$.

- $|\psi_i\rangle \neq |\gamma\rangle \Rightarrow |\psi_j\rangle = |\delta\rangle$
- $|\psi_i\rangle = |\gamma\rangle \neq$ No propagation
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 i & |\gamma\delta\rangle\langle\gamma\delta| & j \\
 \hline
 |\psi_i\rangle & |\psi_j\rangle \\
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\[ \begin{array}{ccc}
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Multi-qubit Propagation

1. Assign $|\psi_i\rangle$ to qubit $i$.
2. Propagate via a breadth first traversal of the constraint graph.
3. Stop if no propagation is possible or a contradiction is found.

Contradiction: When a qubit $j$ is assigned different states while propagating along different paths from $i$ to $j$.

Lemma (Propagation Lemma, Informal Statement)
Let the propagation $(i, |\psi_i\rangle)$ on $G(H)$ extend the assignment to $|\psi_i\rangle \otimes |\Phi\rangle \otimes |\text{rest}\rangle$. If the propagation is:
1. "Unsuccessful", there is no solution of the form $|\psi_i\rangle \otimes |\text{rest}\rangle$.
2. Otherwise, there exists a solution of the form $|\psi_i\rangle \otimes |\Phi\rangle \otimes |\text{rest}\rangle$.
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Algorithm Sketch
Part A: Rank-3 and Rank-1 Product Constraints

\[ 2\text{-QSATSolver}(G(H)) \]

**Step 1** For all rank-3 constraints \( \Pi_{ij} \) in \((G(H))\)

a. Assign the unique state orthogonal to \( \Pi_{ij} \) to qubits \( i, j \).

**Step 2** Propagate the previous assignments and if a contradiction is found return "Unsuccessful".
Part A: Rank-3 and Rank-1 Product Constraints

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**Step 3** For all rank-1 product constraints \(\Pi_{ij} = |\alpha_i\beta_j\rangle\langle\alpha_i\beta_j| \) in \((G(\mathcal{H}))\)

a. Propagate in parallel the assignments \((i, |\alpha_i^\perp\rangle)\) and \((j, |\beta_j^\perp\rangle)\).

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Linear time algorithm for 2QSAT
Part A: Rank-3 and Rank-1 Product Constraints

2-QSAT Solver \( (G(\mathcal{H})) \)

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   a. Assign the unique state orthogonal to $\Pi_{ij}$ to qubits $i, j$.

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Step 3 For all rank-1 product constraints $\Pi_{ij} = |\alpha_i \beta_j\rangle \langle \alpha_i \beta_j|$ in $(G(H))$
   a. Propagate in parallel the assignments $(i, |\alpha_i^\perp\rangle)$ and $(j, |\beta_j^\perp\rangle)$.
   b. If both are unsuccessful, return "Unsuccessful".
   c. Accept the assignments of the first successful propagation.
Part A: Rank-3 and Rank-1 Product Constraints

2-QSATSolver\((G(\mathcal{H}))\)

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   - Accept the assignments of the first successful propagation.

\[\begin{align*}
|\alpha^\perp\rangle \\
i
j \\
\Pi_{j\ell} \\
\psi_k \\
k
\end{align*}\]
Part B: Rank-1 Entangled Constraints

Step 5  Pick an unassigned qubit $i$ and propagate $(i, |0\rangle)$.

Step 6  If qubit $j$ faces a contradiction, there are two paths $p_1, p_2$ from $i$ to $j$.  

Lemma (Sliding Lemma [JWZ11])
Consider a system on 3 qubits $(i, j, k)$ with entangled constraints $\Pi_{ij}$ and $\Pi_{ik}$. Then there is a constraint $\Pi_{jk}$ such that the ground spaces of $\Pi_{ij} + \Pi_{ik}$ and $\Pi_{ij} + \Pi_{jk}$ are identical.
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Step 7 **Slide** along $p_1$ and $p_2$ to obtain constraints $\Pi_{ij,1}$ and $\Pi_{ij,2}$.

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Dealing with Entangled Constraints

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Step 7  **Slide** along $p_1$ and $p_2$ to obtain constraints $\Pi_{ij,1}$ and $\Pi_{ij,2}$.

**Fact (Structure of 2-dim subspace)**

*Any 2-dimensional subspace $V$ of the 2-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product state, which can be found in constant time.*
Dealing with Entangled Constraints

Step 5 Pick an unassigned qubit $i$ and propagate $(i, |0\rangle)$.

Step 6 If qubit $j$ faces a contradiction, there are two paths $p_1, p_2$ from $i$ to $j$.

Step 7 Slide along $p_1$ and $p_2$ to obtain constraints $\Pi_{ij,1}$ and $\Pi_{ij,2}$.

Step 8 Use the product constraint in the space of $\Pi_{ij,1} + \Pi_{ij,2}$ to propagate $(i, |\psi_i\rangle)$ and $(j, |\psi_j\rangle)$ in parallel.

Fact (Structure of 2-dim subspace)

Any 2-dimensional subspace $V$ of the 2-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product state, which can be found in constant time.
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- The sliding is constrained by the length of the cycle as each slide can be done in constant time.
- To deal with cycles of entangled constraints, each edge in the paths $p_1, p_2$ is considered at most 4 times – first propagation + sliding + parallel propagation.
Thanks for your attention!
Cost of performing operations on complex numbers
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**Main Result** (from [dBG])

- Bit complexity of the algorithms is $O((m + n)M(n))$
  where $M(n) :=$ Cost of multiplying two $n$ bit numbers.
- Explicit constructions show that $O(n + m)$ bit complexity is not possible for general 2-QSAT instances.
- When all constraints are product, the bit complexity matches the linear bit complexity of 2SAT