Borel Coalgebra and Non-Wellfounded Logic

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joint work with Francisco Mota

Automata and Logic – A Recurring Theme

- logic and semantics
- algebra and coalgebra
- static vs dynamic
- operational vs denotational

Logic has contributed enormously to computer science.

Can we repay the debt?

Alternating Turing machines [with Chandra & Stockmeyer 1976]

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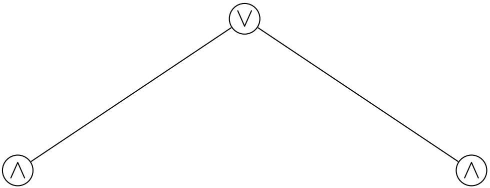

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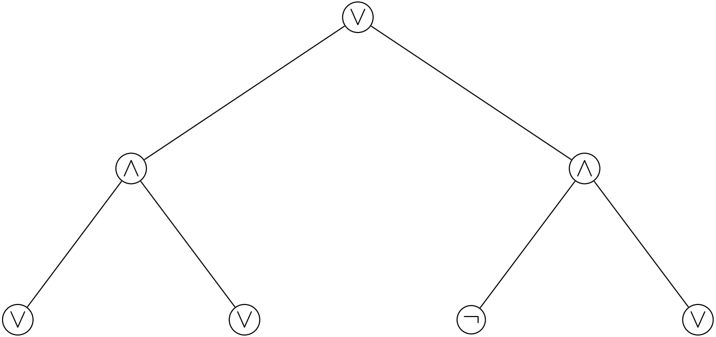

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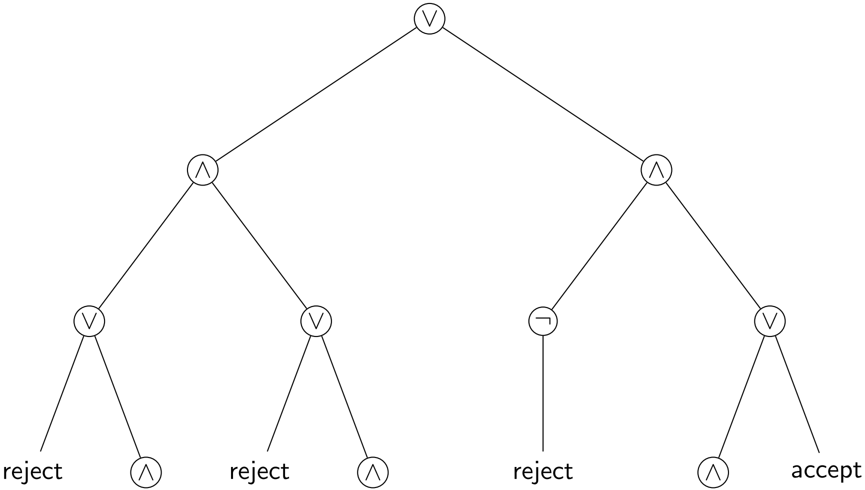

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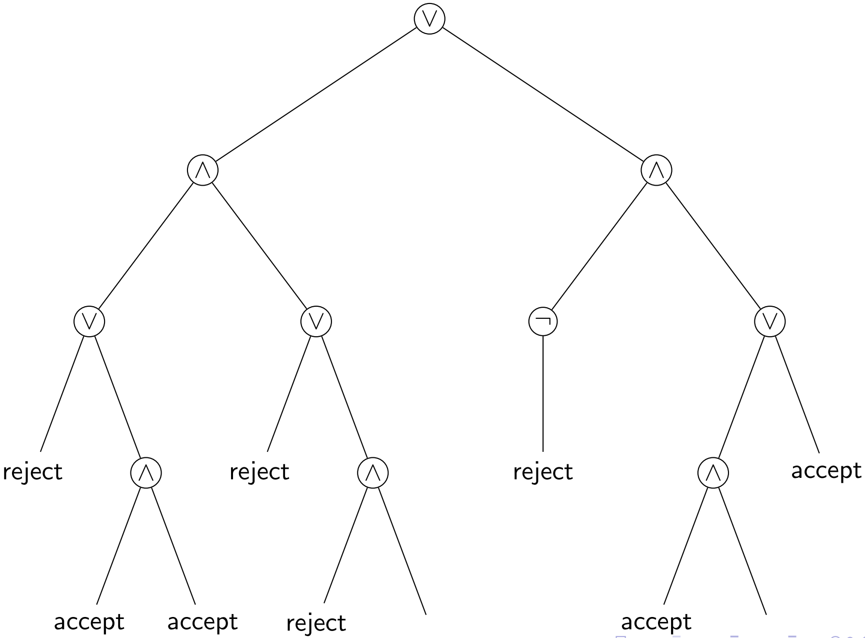

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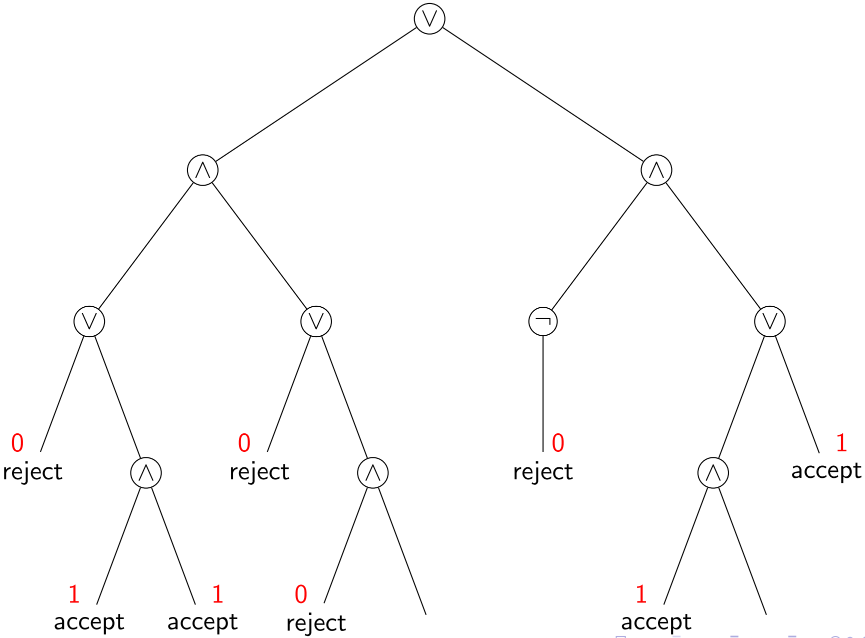

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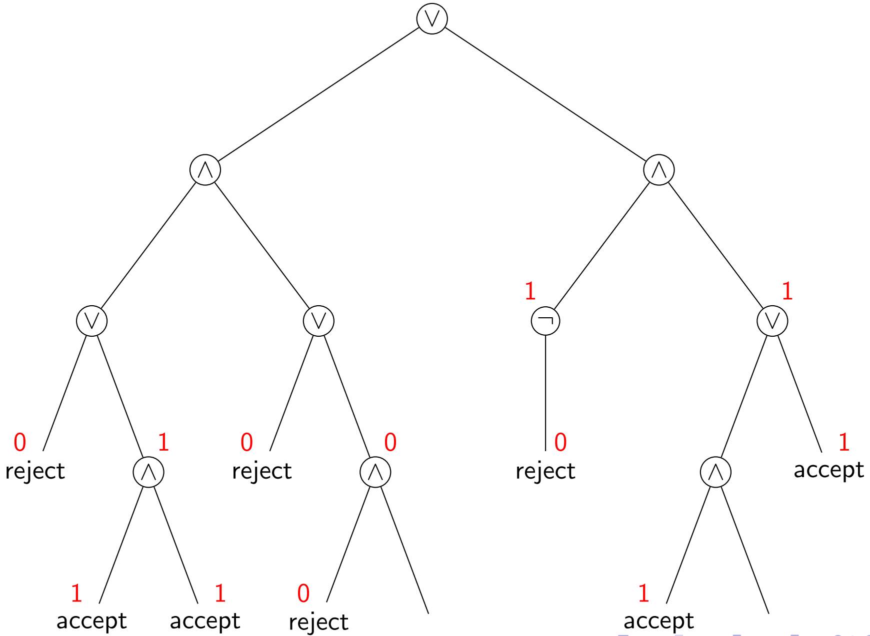

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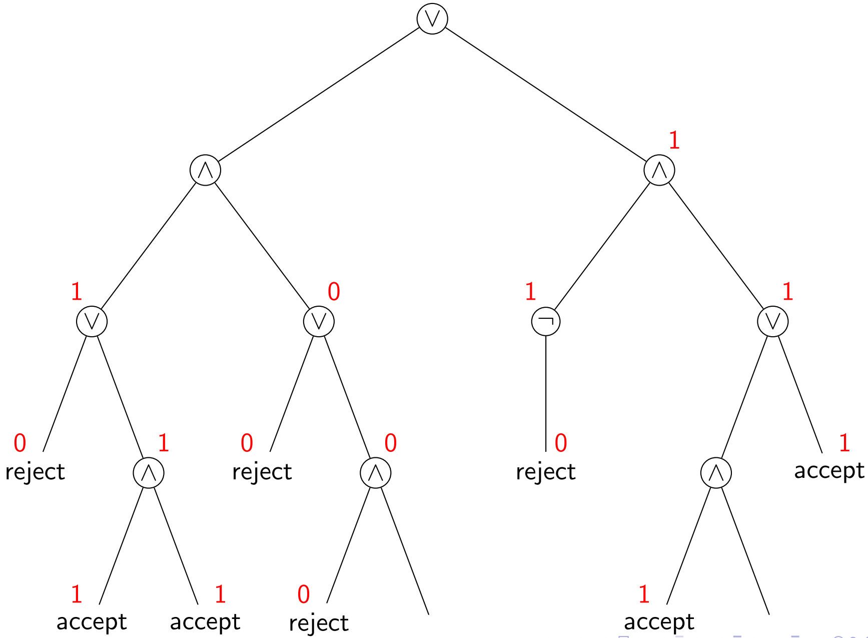

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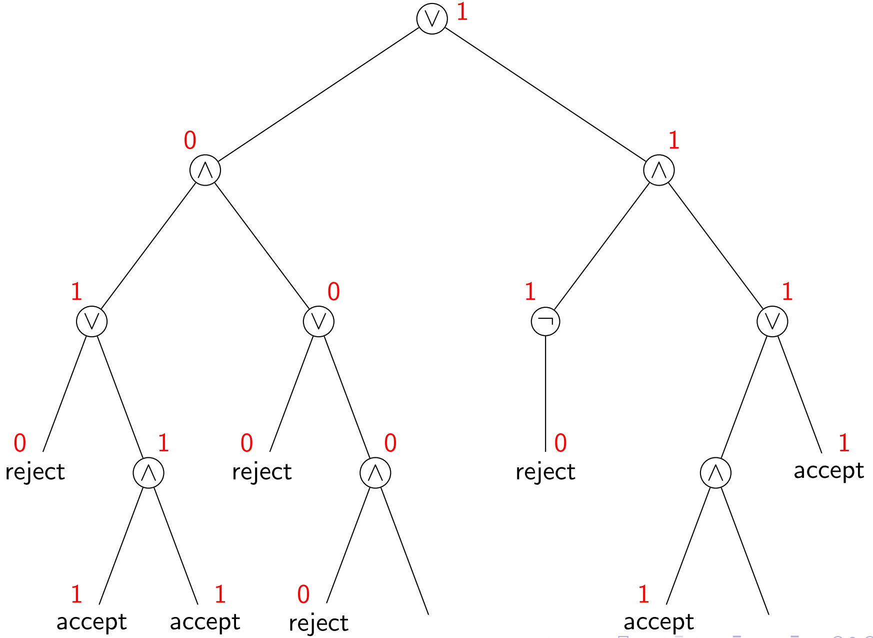

IND Programs [with David Harel 1984]

$\ell : x := \exists$ $\qquad$ $\ell : y := \forall$

$\ell :$ `if` $R(\bar{x})$ `then` ℓ_1 `else` ℓ_2

$\ell :$ `accept` $\qquad$ $\ell :$ `reject`

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Theorem

- *Over any structure, IND programs accept exactly the sets definable by first-order induction.*
- *Over $\mathbb{N}$, IND programs accept exactly the Π^1_1 sets.*

Examples

Transitive closure of a relation R

$$
\begin{array}{ll}
\ell: & \texttt{if } x = y \texttt{ then accept} \\
 & z := \exists \\
 & \texttt{if } \neg R(x,z) \texttt{ then reject} \\
 & x := z \\
 & \texttt{goto } \ell
\end{array}
$$

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Accepts iff $R^*(x, y)$

Examples

Transitive closure of a relation R

```
ℓ :  if x = y then accept
     z := ∃
     if ¬R(x, z) then reject
     x := z
     goto ℓ
```

$x := \exists$
if $x \neq z$ then reject

Accepts iff $R^*(x, y)$

Examples

Winning strategies in games

$$
\begin{array}{ll}
\ell: & y := \exists \\
 & \texttt{if } \neg move(x, y) \texttt{ then reject} \\
 & x := \forall \\
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& \texttt{goto } \ell
\end{array}
$$

Accepts iff first player has a winning strategy from position x

Examples

Well-foundedness of R

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\ell: & y := \forall \\
 & \texttt{if } \neg R(y, x) \texttt{ then accept} \\
 & x := y \\
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\end{array}
$$

Examples

Well-foundedness of R

$$\begin{array}{ll} \ell: & y := \forall \\ & \texttt{if } \neg R(y,x) \texttt{ then accept} \\ & x := y \\ & \texttt{goto } \ell \end{array}$$

Accepts iff R is well-founded below x

Kleene–Suslin Theorem (constructive version)

Theorem (Kleene 1955)

Over $\mathbb{N}$, the inductive relations and the Π^1_1 relations coincide, and the hyperelementary (inductive and coinductive) and Δ^1_1 relations coincide.

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Over $\mathbb{N}$, the inductive relations and the Π^1_1 relations coincide, and the hyperelementary (inductive and coinductive) and Δ^1_1 relations coincide.

Theorem

A relation A is inductive iff $A = L(S)$ for some IND program S.
Moreover, TFAE:

1. *A is inductive and coinductive*
2. *A is hyperelementary*
3. *A is accepted by a well-founded IND program*
4. *A is accepted by an IND program with uniform time bound $< \omega_1^{CK}$*
5. *A is accepted by an IND program that halts on all inputs.*

Other Structures?

Theorem (Barwise et al. 1971, cf. Moschovakis 1969)

Over any countable structure A, the Π^1_1 relations are equivalent to a certain class[1] *of inductively definable relations over $HF(A)$, where $HF(A)$ refers to the structure A augmented with its hereditarily finite sets.*

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[1] Not all inductively definable relations over $HF(A)$ are allowed, but only a certain subclass defined in terms of a restricted form of quantification on sets.

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Theorem (2004)

Over any countable structure, IND programs with dictionaries accept exactly the Π^1_1 relations.

`reset()` `put(x, y)` `containsKey(x)` `get(x)`

Borel Automata [with Francisco Mota 2016]

Fix a set Φ of atomic predicates. A Borel automaton over Φ with states (= statement labels) S is a (possibly infinite) program consisting of labeled statements of the form

$s : \exists A$	$(A \in \wp_{\omega_1}(S))$
$s : \forall A$	$(A \in \wp_{\omega_1}(S))$
$s : \texttt{if } \varphi \texttt{ then } t \texttt{ else } u$	$(\varphi \in \Phi,\ t, u \in S)$
$s : \neg t$	$(t \in S)$

$\texttt{accept} = \forall\varnothing$ $\qquad$ $\texttt{reject} = \exists\varnothing$

Kleene-Suslin Theorem (nonconstructive version)

Borel sets = smallest σ-algebra containing the open sets
analytic sets = projections of Borel sets

Theorem (Suslin 1917)

Let A be a subset of a Polish space. Then A is Borel iff A is both analytic and coanalytic.

Kleene-Suslin Theorem (nonconstructive version)

Borel sets = smallest σ-algebra containing the open sets
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Theorem (Suslin 1917)

Let A be a subset of a Polish space. Then A is Borel iff A is both analytic and coanalytic.

Theorem

Let A be a subset of a Polish space. Then A is coanalytic iff $A = L(S)$ for some Borel automaton S. Moreover, TFAE:

1. *A is analytic and coanalytic*
2. *A is Borel*
3. *A is accepted by a well-founded Borel automaton*
4. *A is accepted by a Borel automaton with uniform time bound $< \omega_1$*
5. *A is accepted by a Borel automaton that halts on all inputs.*

Thanks!