Robust Assignments via Ear Decompositions and Randomized Rounding

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1. Problem definition and applications

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   - $O(1)$-Approximation for the unweighted version
   - The simplest non-trivial case

Problem definition and applications

Robust Assignment Problem

Certain problem

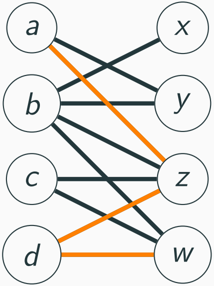
- Given $G = (R \dot{\cup} T, E)$ with $|T| \leq |R|$, costs $c \in \mathbf{R}^E_{\geq 0}$
- Find a minimum-cost assignment of G, i.e. a set of non-adjacent edges from G that covers all nodes in T

Robust version

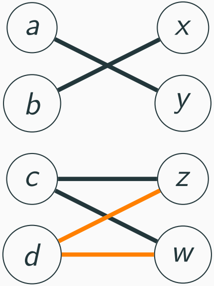
- Uncertainty set $F := \{f_1, \ldots, f_k\} \subseteq E$ (vulnerable edges)
- If scenario f_i emerges, then edge f_i is deleted from G

$$\begin{aligned} \min \quad & \textstyle\sum_{e \in X} c_e \\ \text{s.t.} \quad & \forall f \in F : X \setminus \{f\} \text{ contains} \\ & \text{an assignment of } G \end{aligned} \qquad (\text{RAP})$$

Example (unit weights)



$F = \{\{a, z\}, \{d, z\}, \{d, w\}\}$



optimal solution

Classification within Robust Optimization

- RAP fits into the subclass of redundancy-based robust optimization problems, e.g.
  - Minimum k-edge-connected Spanning Subgraph Problem *(e.g. [Gabow et al., 2005], [Sebő and Vygen, 2014])*
  - Robust Facility Location Problem *([Jain and Vazirani, 2000], [Swamy and Shmoys, 2003])*
- RAP is a bulk-robust problem *[Adjiashvili et al., 2015]*)

Possible areas of application

Bulk-robustness is applicable to real-world long-term decision problems where “infrastructure” must be fixed in advance.

In our setting,

- stable client to service provider relationships
- robust long-term schedules and staff training,
- manufacturing process flexibility

In the following

Let $G = (R \dot\cup T, E)$, $c \in \mathbf{R}^E_{\geq 0}$ and $F \subseteq E$ be a RAP instance.

- the graph G is balanced, i.e. $|R| = |T|$ and the assignments become perfect matchings
- every instance is feasible

Algorithmic results

How hard is the problem?

Theorem

Set Cover can be equivalently restated as RAP, preserving the cost.

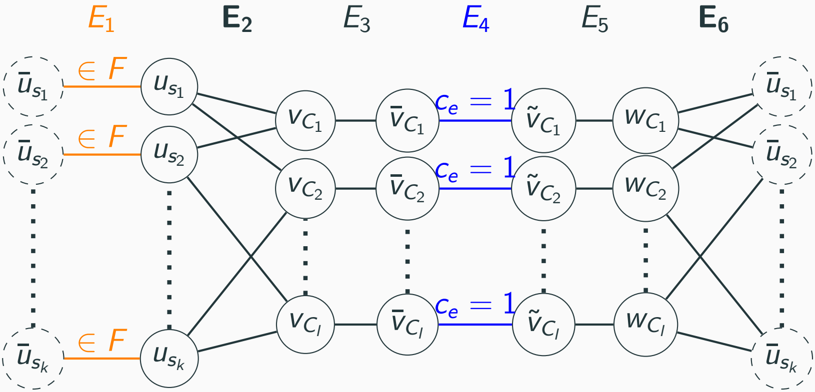
Theorem ([Feige, 1998])

For any $d < 1$, there is no $d \cdot \log n$ –approximation for the Set Cover problem, unless NP $\subseteq$ DTIME$(n^{\log \log n})$.

How hard is the problem?

Proof idea:

Reduce a Set Cover instance $(\{s_1, \dots, s_k\}, \{C_1, \dots, C_l\})$ to the following RAP instance with a graph on 4k + 2l nodes:



Asymptotically tight randomized algorithm

Theorem

RAP admits a randomized polynomial $O(\log n)$-approximation.

Main technical ingredients:
matching-covered graphs and randomized rounding.

Technical ingredients I

Definition

A graph $G = (V, E)$ is called matching-covered if

each edge $e \in G$ is contained in some perfect matching of G

Why is it useful?

Up to technicalities, inclusion-wise minimal feasible solutions to RAP are matching-covered and vice versa.

Technical ingredients II

We use the following IP formulation for RAP:
(P_G is the perfect matching polytope for the graph G)

$$\begin{array}{llcll}
\min & c^\top y & & & \\
\text{s.t.} & x^{-f} & \in & P_G, & \text{for each } f \in F, \\
& x_f^{-f} & = & 0, & \text{for each } f \in F, \\
& y & \geq & x^{-f}, & \text{for each } f \in F, \\
& x^{-f} & \in & \{0,1\}^E, & \text{for each } f \in F, \\
& y & \in & \{0,1\}^E &
\end{array}$$

$O(\log n)$-Approximation for RAP

$(x, y) \leftarrow$ optimal solution of the LP relaxation
$X \leftarrow \emptyset$
while X is infeasible **do**
  Select $f \in F$ such that $X \setminus \{f\}$ has no perf. matching
  Decompose x^{-f} into p. matchings: $x^{-f} = \sum_{i=1}^{k} \lambda_i \chi^{M_i^{-f}}$
  Select $\bar{M} \in \{M_i^{-f} \mid i \in [k]\}$ with probability λ_i
  Augment X using edges from $\bar{M}$ connecting distinct connected components in $(V[G], X)$
return X

$O(1)$-Approximation for minimum-cardinality RAP

We now focus on the version of RAP with unit weights, i.e. $c_e = 1$, $\forall e \in E$ (card-RAP).

Theorem

card-RAP admits a $O(1)$-approximation algorithm.

Main technical ingredient:
ear decompositions of matching-covered graphs.

Bipartite ear decomposition

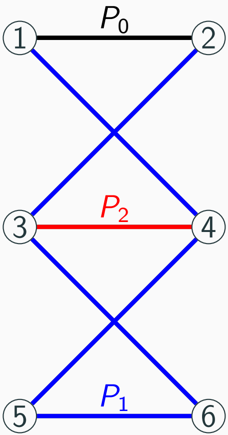
Definition

Consider a bipartite graph H and a subgraph $H' \subseteq H$.

An odd ear of H relative to H' is an odd path P in H connecting nodes u, v in H' such that only u and v are in H'.

Bipartite Ear Decomposition

If H can be constructed by starting with an edge and adding odd ears, then H has a Bipartite Ear Decomposition.
We write $H = P_0 + P_1 + \cdots + P_r$.



A characterization of bipartite m.-c. graphs

Theorem. **(see [Lovász and Plummer, 1986])**

A bipartite graph H is matching-covered if and only if H has a Bipartite Ear Decomposition.

Observation

Removing ears of length one from a Bipartite Ear Decomposition yields a matching-covered graph.

$O(1)$–Approximation algorithm for card-RAP

Remove all dispensable edges from G

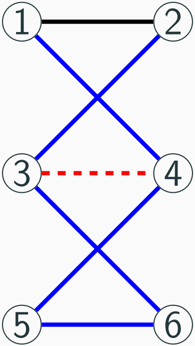
Compute an ear decomposition $G = P_0 + P_1 + \cdots + P_r$

$X \leftarrow E(P_0) \cup \left(\bigcup \{ E(P_i) \colon |E(P_i)| > 1,\ i = 1, \ldots, r \} \right)$

return X

- if $F = E$, then the approximation ratio is 1.5
- if $F \subseteq E$, then the approximation ratio is 3

$O(1)$-Approximation algorithm: Example



Hardness of approximation for card-RAP

Theorem

There is no PTAS for card-RAP, unless P = NP. This is even true for $F = E$.

Proof idea

- [Alimonti and Kann, 1997] showed that, unless P = NP, there is a $\delta > 1$ such that Vertex-Cover Problem in sub-cubic graphs (VC 3) does not admit a polynomial δ-approximation.
- Restate VC 3 in terms of a set cover problem, and use an extended reduction from above. □

The simplest non-trivial case of card-RAP

How hard is the most elementary, non-trivial case of card-RAP (two vulnerable edges, i.e. $F = \{f_1, f_2\}$)?

Theorem

card-RAP is NP-hard with only two vulnerable edges.

Remark

To the best of our knowledge, this is the first example of an NP-hard robust counterpart of a polytime optimization problem with a constant number of vulnerable resources.

Questions?