Linear time algorithm for Quantum 2SAT

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This work done jointly with:

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Background

Classical 2SAT

A classical 2SAT instance Φ is a Boolean formula defined on

- a set of n variables: $\{x_1, \ldots, x_n\}$
- as a conjunction of m clauses: $\{C_1, \ldots, C_n\}$ where
- each clause is an `OR` of at most 2 literals (i.e. x_i and $\overline{x}_i$)

Goal: Find an assignment to the variables so that Φ evaluates to `true`.

Example (2SAT)

An instance: $\Phi = (x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_2) \wedge (x_1 \vee \overline{x}_4) \wedge (x_4) \wedge (x_2 \vee x_3)$
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Algorithms for 2SAT

1. Even, Itai & Shamir (1976): A backtracking, resolution based search of possible assignments.
2. Apsvall, Plass & Tarjan (1979): Finds strongly connected components in the implication graph of the instance.

Both algorithms have an optimal $O(n+m)$ running time.

Quantum 2SAT (2-QSAT)

A quantum 2SAT instance $\mathcal{H}$ is a 2-local Hamiltonian defined on

- n qubits: $\{x_1, \ldots, x_n\}$
- as a sum of m local terms: $\mathcal{H} = \sum_{uv} \Pi_{uv}$ where
- each Π_{uv} is a projector acting non-trivially on qubits (u, v);

Example (Q2SAT)

A 2-QSAT *instance:* $\mathcal{H} = \Pi_{12} + \Pi_{23} + \Pi_{34}$ *with*

$\Pi_{12} = |00\rangle\langle 00| + |11\rangle\langle 11|;$ $\Pi_{34} = |\Psi^-\rangle\langle\Psi^-|;$ $\Pi_{23} = |01\rangle\langle 01|;$

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- each Π_{uv} is a projector acting non-trivially on qubits (u, v);
- the smallest eigenvalue of $\mathcal{H}$ is its ground energy and
- the corresponding eigenvector $|\psi\rangle$ is the ground state of $\mathcal{H}$.

Goal: Given $\mathcal{H}$, output a ground state if the ground energy is 0 or "Unsatisfiable" otherwise.

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Solving 2-QSAT

Prior Work

An $O(n^4)$ 2-QSAT algorithm by Bravyi (2006) based on finding the transitive closure of a directed graph.

- For $k \geq 3$, k-QSAT is QMA_1-complete

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Related Work

- Quantum analogue of the APT algorithm [dBG]
  - Uses the notion of transfer matrices to mirror the implications in Boolean Formulae.

Improvements in run-time

Bravyi's algorithm uses the following approach:

- For every qubit triple (i, j, k), if there is a constraint on (i, j) and (j, k), add an implied constraint acting on (i, k).
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The linear time algorithms approach an instance by:

- Analyzing only a part of the instance and manipulating it with local operations.
- Local sections of the instance on being solved are decoupled from the rest of instance.
- Governed by graph traversals that can be executed in linear time.

Algorithm Building Blocks

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Assume WLOG that $\mathcal{H}$ contains rank-1 and rank-3 projectors only.

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- n vertices representing each qubit and
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Example (Constraint Graph)

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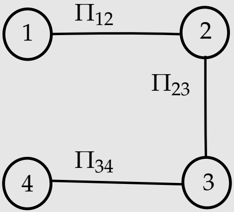

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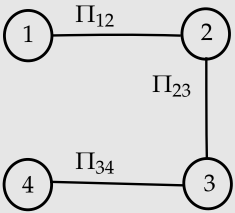

Theorem (Product State Theorem [CCD+11,ASSZ15])

Any satisfiable 2-QSAT *instance has a ground state which is a tensor product of one qubit and two-qubit states, where two-qubit states only appear in the support of rank-3 projectors.*

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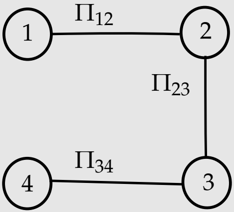
$$\textit{Ground State} = |\Psi^+\rangle_{12} \otimes |0\rangle_3 \otimes |1\rangle_4$$



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Let $\Pi_{ij} = |\psi\rangle\langle\psi|$ *be a rank-*1 *projector and* $|\alpha\rangle$ *be the state assigned to* i*. Then,* Π_{ij} *propagates* $|\alpha\rangle$ *if, up to a phase, there exists a unique* 1*-qubit state* $|\beta\rangle$ *such that* $\langle\psi|(|\alpha\rangle_i \otimes |\beta\rangle_j) = 0$*.*

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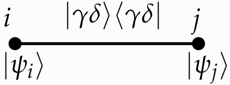

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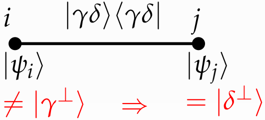

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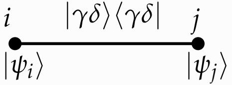

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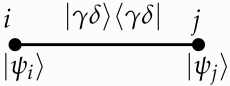

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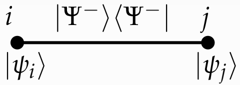

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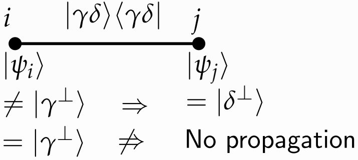

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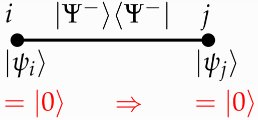

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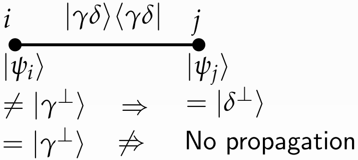

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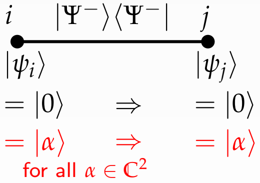

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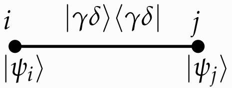

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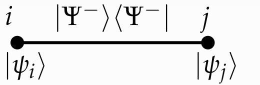

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A product constraint will not propagate a state when already satisfied.

An entangled constraint always propagates every state.

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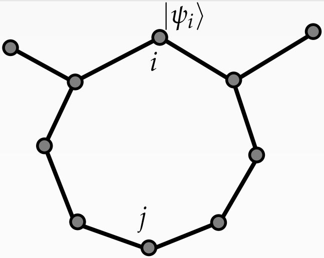

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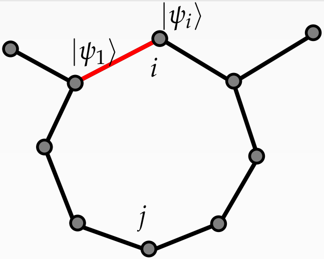

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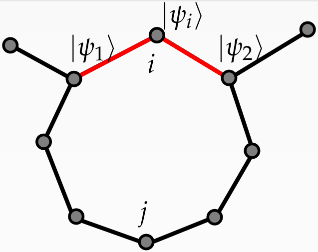

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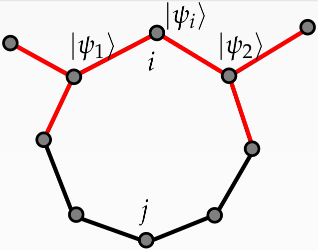

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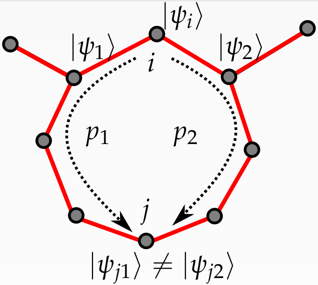

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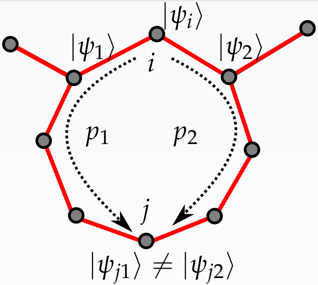

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Lemma (Propagation Lemma, Informal Statement)

Let the propagation $(i, |\psi_i\rangle)$ on $G(\mathcal{H})$ extend the assignment to $|\psi_i\rangle \otimes |\Phi\rangle$. If the propagation is:

1. *"Unsuccessful", there is no solution of the form $|\psi_i\rangle \otimes |rest\rangle$.*
2. *Otherwise, there exists a solution of the form $|\psi_i\rangle \otimes |\Phi\rangle \otimes |rest\rangle$.*

Algorithm Sketch

Part A: Rank-3 and Rank-1 Product Constraints

2-QSAT`Solver`$(G(\mathcal{H}))$

Step 1 For all rank-3 constraints Π_{ij} in $(G(\mathcal{H}))$

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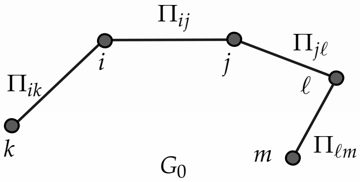

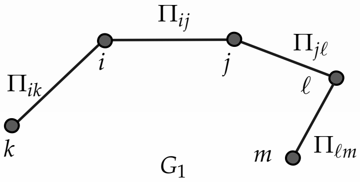

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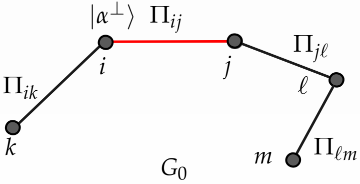
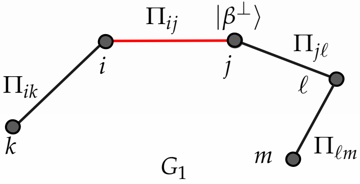




Part A: Rank-3 and Rank-1 Product Constraints

2-QSAT`Solver`$(G(\mathcal{H}))$

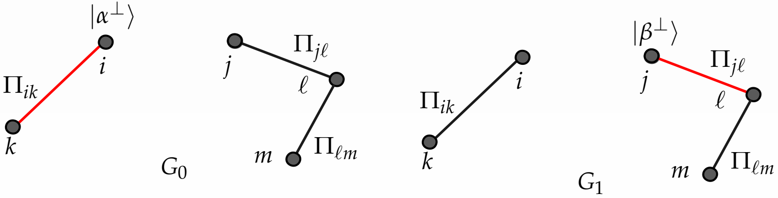
Step 1 For all rank-3 constraints Π_{ij} in $(G(\mathcal{H}))$

- a. Assign the unique state orthogonal to Π_{ij} to qubits i, j.

Step 2 Propagate the previous assignments and if a contradiction is found return "Unsuccessful".

Step 3 For all rank-1 product constraints $\Pi_{ij} = |\alpha_i\beta_j\rangle\langle\alpha_i\beta_j|$ in $(G(\mathcal{H}))$

- a. Propagate in parallel the assignments $(i, |\alpha_i^{\perp}\rangle)$ and $(j, |\beta_j^{\perp}\rangle)$.



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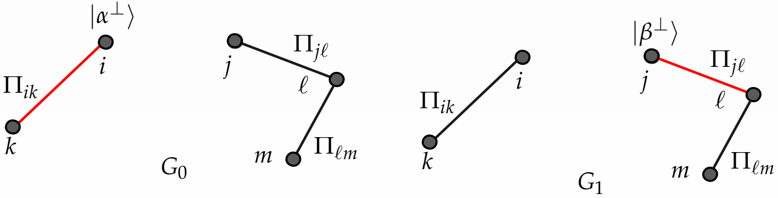

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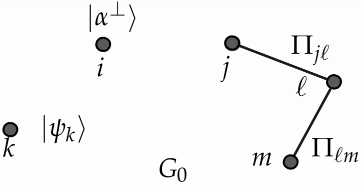

Part B: Rank-1 Entangled Constraints

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Lemma (Sliding Lemma [JWZ11])

Consider a system on 3 qubits (i, j, k) with entangled constraints Π_{ij} and Π_{ik}. Then there is a constraint Π_{jk} such that the ground spaces of $\Pi_{ij} + \Pi_{ik}$ and $\Pi_{ij} + \Pi_{jk}$ are identical.

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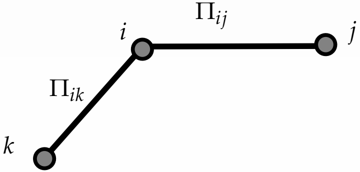

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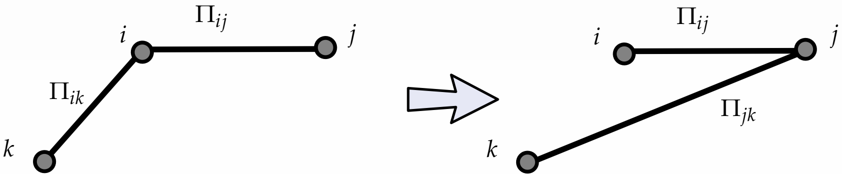

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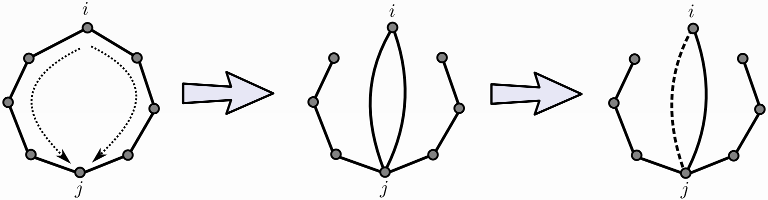

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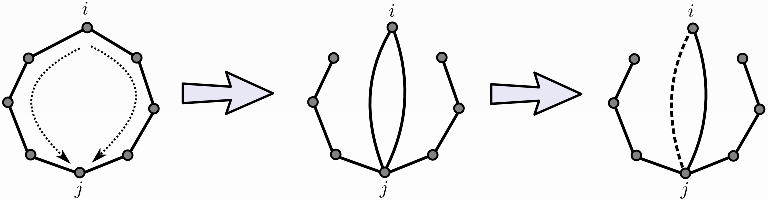

Fact (Structure of 2-dim subspace)

Any 2-dimensional subspace V of the 2-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ contains at least one product state, which can be found in constant time.

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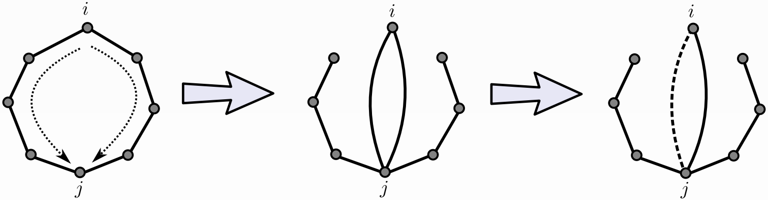
Step 8 Use the product constraint in the space of $\Pi_{ij,1} + \Pi_{ij,2}$ to propagate $(i, |\psi_i\rangle)$ and $(j, |\psi_j\rangle)$ in parallel.



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- The sliding is constrained by the length of the cycle as each slide can be done in constant time.
- To deal with cycles of entangled constraints, each edge in the paths p_1, p_2 is considered at most 4 times – first propagation $+$ sliding $+$ parallel propagation.

Thanks for your attention!

Cost of performing operations on complex numbers

Bit Complexity

Bit complexity refers to the number of bit-wise operations performed in the course of an algorithm.

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Main Result (from [dBG])

- Bit complexity of the algorithms is $O((m+n)M(n))$

  where $M(n) :=$ Cost of multiplying two n bit numbers.
- Explicit constructions show that $O(n+m)$ bit complexity is not possible for general 2-QSAT instances.
- When all constraints are product, the bit complexity matches the linear bit complexity of 2SAT