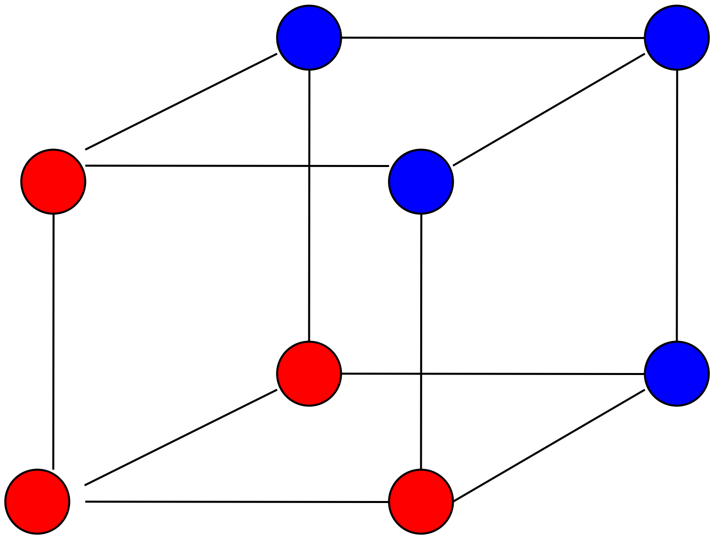
On The Sensitivity Conjecture



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The Sensitivity Conjecture: Believe It or Not?

Sensitivity of Boolean Functions

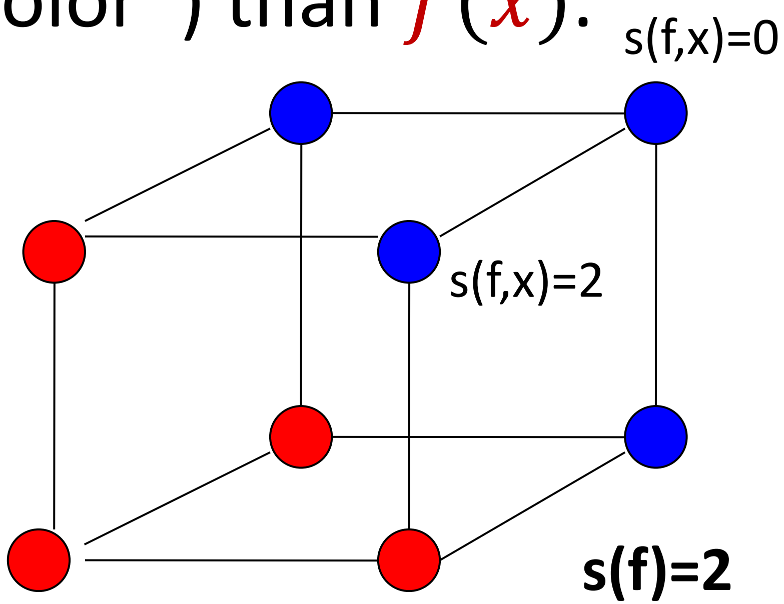
Let $f: \{0,1\}^n \to \{0,1\}$ be a Boolean function.

The sensitivity of f at $x \in \{0,1\}^n$:

$s(f,x)$ = # of **neighbors** of x in Hamming cube with different f-value ("color") than $f(x)$.

The sensitivity of f:

$$s(f) = \max_x s(f,x)$$



Block Sensitivity of Boolean Functions

Let $f:\{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function

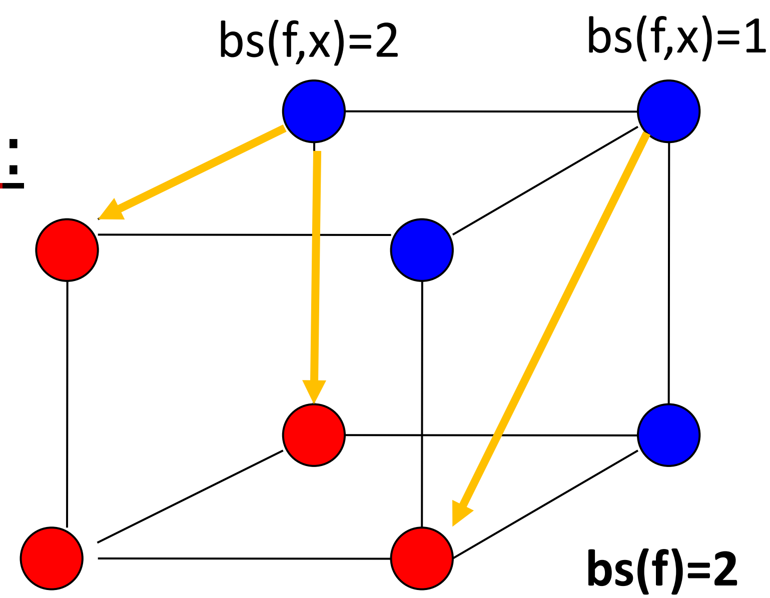
The block sensitivity (bs) of f at $x \in \{0,1\}^n$:

$bs(f,x)$ = maximal number of **disjoint blocks** $B_1, B_2, \ldots, \subseteq [n]$ with $f(x) \neq f(x^{B_i})$.

The block sensitivity (bs) of f:

$$bs(f) = \max_x bs(f,x)$$

Obs: $\forall f: s(f) \leq bs(f)$



The Sensitivity Conjecture

Conjecture [Nisan-Szegedy'92]: $\exists$ a constant c:

$\forall$ Boolean functions f, $bs(f) \le s(f)^c$.

More boldly, $\forall f: bs(f) \le s(f)^{\mathbf{2}}$

Known upper bounds **[Simon, KK, ABGMSZ, APV]**

$$bs(f) \le 2^{s(f)} \cdot s(f)$$

Known lower bounds **[Rubinstein, Virza, AS]**

$$\exists f: bs(f) \ge \Omega(s(f)^{\mathbf{2}})$$

Other Complexity Measures

- **D(f)** – **Deterministic decision tree complexity**
- **R(f)** – **Probabilistic decision tree complexity**
- **Q(f)** – **Quantum decision tree complexity**
- **C(f)** – **Certificate complexity**
- **deg(f)** – **Real degree**
- **deg_{∞}(f)** – **L_{∞} approximate degree**
- **bs(f)** – **Block sensitivity**

[Blum-Impagliazzo, Tardos, Hartmanis, Nisan, Nisan-Szegedy, BBCMW…]: all parameters are polynomially related.

The sensitivity conjecture is equivalent to $M(f) \leq s(f)^{O(1)}$ for any complexity measure M above.

Kenyon-Kutin’s Approach

The ℓ-block sensitivity of f at $x \in \{0,1\}^n$:

bs_ℓ(f,x) = maximal number of **disjoint blocks** $B_1, B_2 \ldots, \subseteq [n]$, each of size at most ℓ, with $f(x) \neq f(x^{B_i})$.

Observation: WLOG $\ell \leq s(f)$.

Theorem [Kenyon-Kutin’04]: $2 \leq \ell \leq s(f)$

1. $bs_\ell(f) \leq \frac{e}{\ell} \cdot s(f) \cdot bs_{\ell-1}(f)$
2. $bs_\ell(f) \leq \frac{e}{(\ell-1)!} \cdot s(f)^\ell$

Can we improve these inequalities?

For $\ell = s(f)$ this gives $bs(f) = bs_\ell(f) \leq \tilde{O}\left(e^{s(f)}\right)$

Kenyon-Kutin's Approach – Small ℓ

$\ell = 2$:

[KK, **T**]: $bs_2(f) \le \frac{e}{2} \cdot s(f)^2$

Nearly Tight

[Rubinstein, Virza, AS]: $\exists f: bs_2(f) \ge \frac{2}{3} \cdot s(f)^2$

$\ell = 3$:

[KK]: $bs_3(f) \le O(s(f)^3)$

[Rubinstein, Virza, AS]: $\exists f: bs_3(f) \ge \frac{2}{3} \cdot s(f)^2$

What is the right answer?

Understanding $bs_3(f)$ is important!

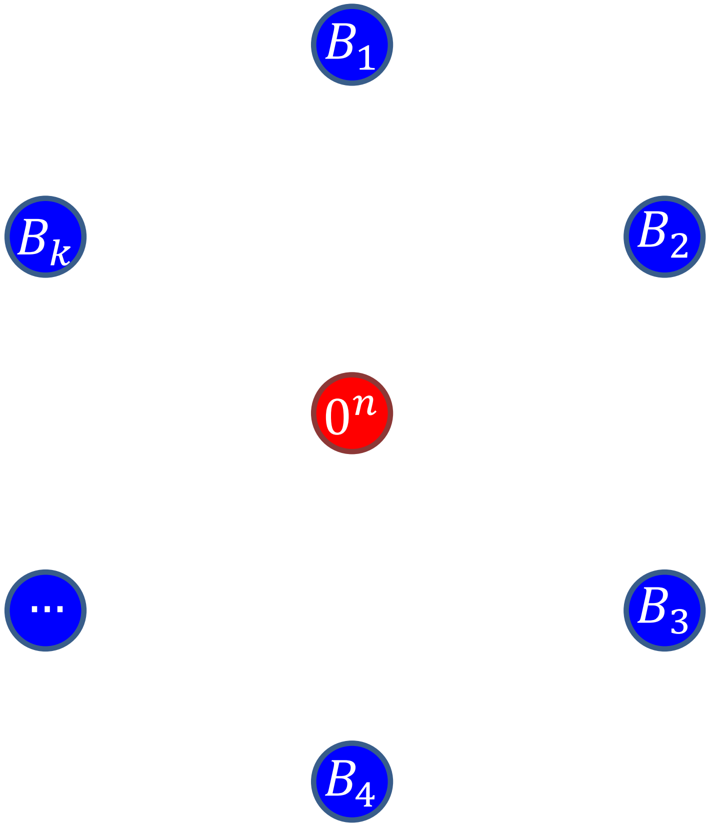
Improving upper / lower bounds for $bs_{\mathbf{3}}$ vs. s

➔ better bounds on bs vs. s

Claim[Folklore]: If $\exists g$: $bs_3(g) \geq s(g)^{2+\epsilon}$,
then $\exists$ inf. many f's with $bs(f) \geq s(f)^{2+\epsilon}$.

Thm1: If $\forall f$: $bs_{\mathbf{3}}(f) < s(f)^{2.999}$,
then $\forall f$: $bs(f) < 2^{o(s(f))}$.

New Proof for Kenyon-Kutin



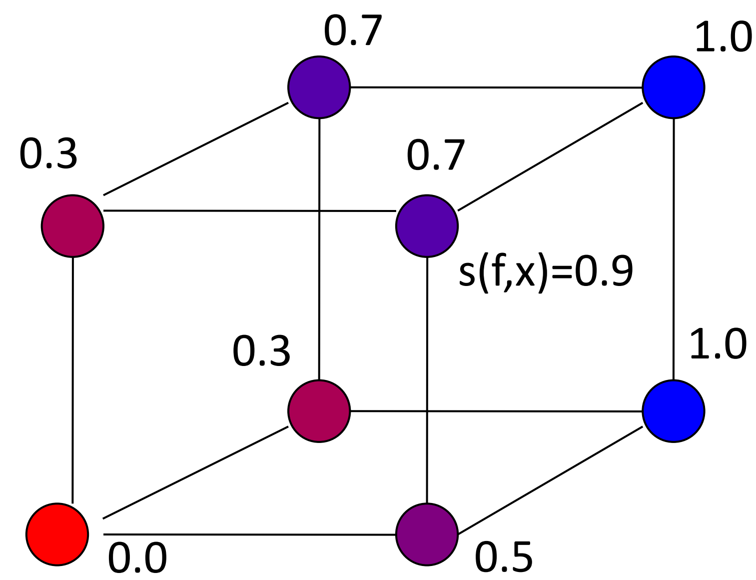
Generalization to Bounded functions

Let $f:\{0,1\}^n \rightarrow [\mathbf{0},\mathbf{1}]$

The sens. of f at $x \in \{0,1\}^n$:

$s(f,x) = \sum_{y\sim x} |f(x) - f(y)|$

Similarly, generalize bs, bs_ℓ



New proof for Kenyon-Kutin's result works for Bounded functions

Reason: most points in a ball centered at x_0 have f-value **close** to $f(x_0)$.

However, in this case the bounds are tight!

The Sensitivity Conjecture is False for Bounded Functions

Thm3: For all ℓ, n, $\exists$ a function $f: \{0,1\}^n \to [0,1]$ with

1. $bs_\ell(f) \geq \left\lfloor \frac{n}{\ell} \right\rfloor$
2. $s(f) = O(\ell \cdot n^{1/\ell})$

In particular, for $\ell = \log(n)$, an exponential separation $s(f) = O(\log n)$ and $bs(f) \geq \Omega(n/\log n)$.

Sensitivity vs. Decision Tree

- First super-quadratic separation between **decision-tree complexity** and **sensitivity**
- Separation based on a "gadget" $f: \{0,1\}^{42} \to \{0,1\}$ with $s(f) = 6, D(f) = 42$ composed with itself many times.
- $s(f^k) = 6^k$
- $D(f^k) = 42^k \approx s(f^k)^{2.086}$
- More examples found using computer search
- [Ben-David'16]: cubic separations

$$\exists f:\ D(f) = \widetilde{\Omega}(s(f)^3)$$

The Sensitivity Conjecture: Believe It or Not?

Why should you believe?

- It stood for along time without a **refutation**.
- Best lower bounds: $bs(f) = \Omega(s(f)^2)$ for more than **20** years.
- Conj holds for families of Boolean functions: **monotone**, **symmetric** [Nis]
- **Consequences** of the conjecture were proved, e.g. any low-sensitivity function is in **NC1** [GNSTW'16].

Why should you doubt?

- It stood for along time without a **proof**.
- Best upper bounds: $bs(f) = \exp(O(s(f)))$ for more than **30** years.
- Don't know **how many** small-sensitivity functions are there? [GNSTW'16] For s = log n, this number can be between $\exp(n)$ to $\exp(n^{\log n})$
- **Can't** even **brute-force** to check if $\exists f$ with bs(f)=26 and s(f)=5.
- Computer-search found D(f) vs s(f) separations **we weren't aware of.**

Thank You!

Thm3 [Case $\ell = 3$]: $\exists$ a function $f: \{0,1\}^n \to [0,1]$ with $bs_3(f) \geq n/3$ & $s(f) = O(n^{1/3})$.

Construction: Partition [n] to $n/3$ blocks of size 3:
$\{\{1,2,3\}, \{4,5,6\}, \dots, \{n-2, n-1, n\}\}$

Let $x \in \{0,1\}^n$ and $H_i(x)$ be the hamming weight of x on the i-th block.

$$w_i(x) = \begin{cases} 0, & H_i(x) = 0 \\ n^{-2/3}, & H_i(x) = 1 \\ n^{-1/3}, & H_i(x) = 2 \\ 1, & H_i(x) = 3 \end{cases}$$

$$f(x) = \min(1, \textstyle\sum_i w_i(x)).$$

Proof: $bs(f, 0^n) \geq n/3$: **Easy!**

Key Point for $s(f)$=O($n^{1/3}$): If $\sum_i w_i(x) \geq 2$, $s(f,x) = 0$.

➔Enough to consider x's with $\sum_i w_i(x) < 2$:

$\#\{i : H_i(x) = 0\} \leq n$,	each may contribute $n^{-2/3}$ to s(f,x)
$\#\{i : H_i(x) = 1\} \leq O(n^{2/3})$,	each may contribute $n^{-1/3}$ to s(f,x)
$\#\{i : H_i(x) = 2{,}3\} \leq O(n^{1/3})$,	each may contribute 1 to s(f,x)