Graph Minors for Preserving Terminal Distances Approximately - Lower and Upper Bounds

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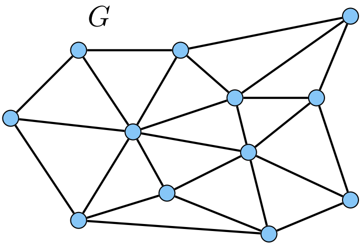
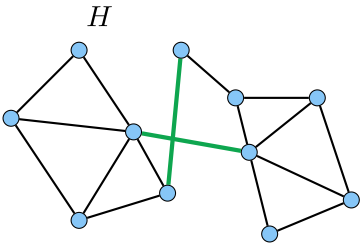
Graph Sparsification





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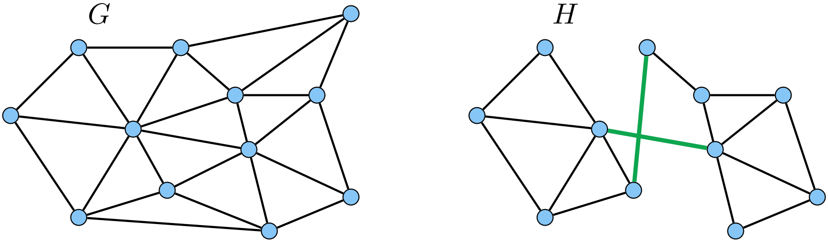

- compute a smaller graph that preserves some **feature/property** of the input graph.

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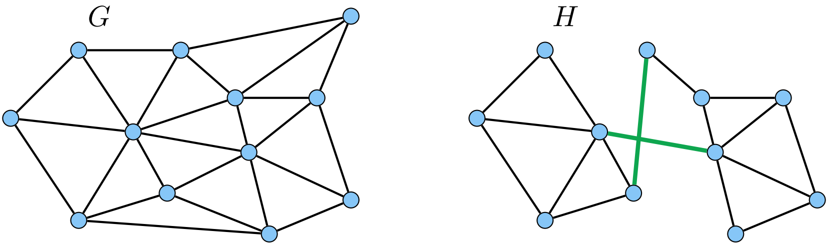

- compute a smaller graph that preserves some **feature/property** of the input graph.
- typical examples: **Cut/Spectral Sparsifiers**, **Spanners**.

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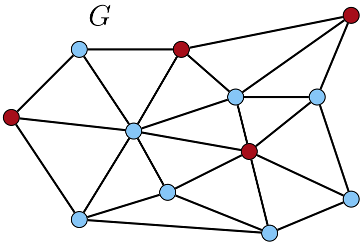
This talk: Vertex Sparsification

Given:

- weighted graph G
- a subset of k vertices T, called **terminals**



This talk: Vertex Sparsification

Given:

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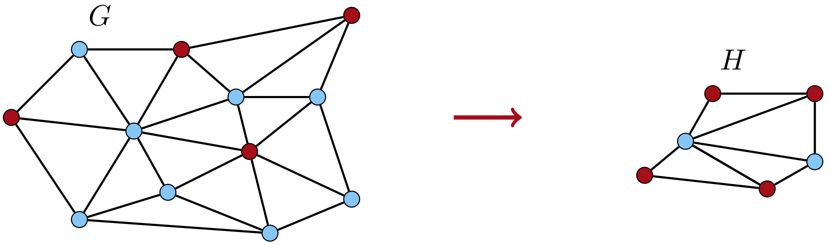
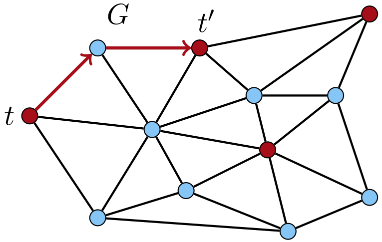
Compute:

- small graph H that contains T & preserves some **feature** of G.



Terminal Distances

- Graph $G = (V, E)$ with edge lengths $\ell : E \to \mathbb{R}_+$– very huge
- k terminals $T \subset V$ – usually small



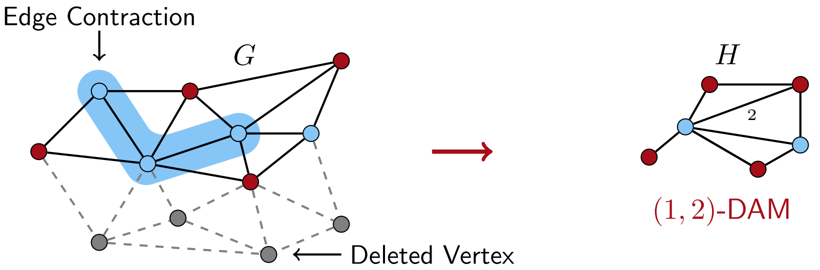
- we are interested only on **terminal distances**
- $d_G(t, t')$ denotes the shortest-path induced by ℓ in G.

Distance Approximating Minor

- A graph $H = (V', E', \ell')$ with $T \subset V'$ is an **α-distance approximating minor** (α-DAM) of G if H is a **minor** of G and:

$$\forall t, t' \in T, \quad d_G(t,t') \le d_H(t,t') \le \alpha \cdot d_G(t,t')$$

- H is an $(\alpha, f(k))$-DAM of G if H contains $\le f(k)$ non-terminals.



Previous Work

- Steiner Point Removal Problem – SPR ($(\alpha, 0)$-DAM)
  - [Gupta SODA'01] Trees admit a $(8, 0)$-DAM.
  - [Chan, Xia, Konjevod, Richa APPROX'06]
    The distortion of 8 is optimal for trees.
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- Distance Preserving Minors ($(1, f(k))$-DAM)
  - [Krauthgamer, Nguyen, Zondiner SIDMA14]
    General graphs admit a $(1, \mathcal{O}(k^4))$-DAM.
  - For planar graphs and $\alpha = 1$ we need at least $\Omega(k^2)$ non-terminals.

Our results

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Lower Bounds

- For all $\varepsilon > 0$, there is a family of graphs with k-terminals that do not have a $(2-\varepsilon, k^2/7)$-DAM.
- **Super-linear** guarantees for larger distortions, e.g., $(2.5-\varepsilon, c_1 k^{5/4})$, $(3-\varepsilon, c_2 k^{6/5})$ and more trade-offs in between.

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Upper Bounds

- General: for all $q \geq 1$, $(2q-1, \mathcal{O}(k^{2+2/q}))$-DAM.
- **Planar**: $(1+\varepsilon, \mathcal{O}((k/\varepsilon)^2 \log^2 k)$-DAM.
- Excluded Fixed Minor: $(\mathcal{O}(1), \widetilde{\mathcal{O}}(k^2)$-DAM – improved to a $(1+\varepsilon)$-guarantee by Gupta and DiRenzo.

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- Idea: Make use of the existing lower bounds for the SPR problem.
- Challenge: We need a meaningful way to bring these two seemingly related problems together.
- Our Solution: We present a **black-box** reduction to translate lower-bounds from SPR to DAMs with additional non-terminals.
- Main tool: **Steiner Systems** from Combinatorics!

Recipe for Proving Lower Bounds

1. Take your favourite SPR lower bound and its instance G^*.
2. Compute a Steiner System depending on the size of G^*.
3. Embed multiple copies of G^* into an (almost) bipartite graph G constructed from the Steiner System.
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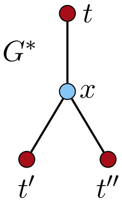
Rest of this talk:

- Demonstrate the simplest application of this reduction.
- State some natural open questions.

SPR lower bound on stars

Lemma (1) [Gupta '01]

- For a star G^* with 3 terminals as leaves, every minor H^* has distortion at least 2.



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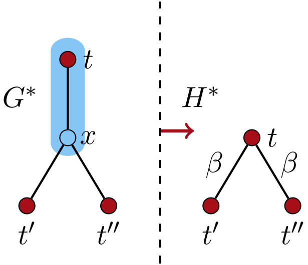

- Minor of G^* is unique up to relabelling of the terminals.
- $\beta = 2$ is optimal for H^*.
- Then $d_{H^*}(t', t'') = 4$, but $d_{G^*}(t', t'') = 2$.

Steiner Triple System

- Recall that T is the set of k terminals.
- An $(3,2)$-**Steiner system** ($(3,2)$-SS) of T is a collection of 3-subsets of T, denoted by $\mathcal{T}$, such that every 2-subset of T is contained in **exactly one** of the 3-subsets.

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- Example: Let $T = \{1, \ldots, 7\}$. The following is a $(3,2)$-SS for T

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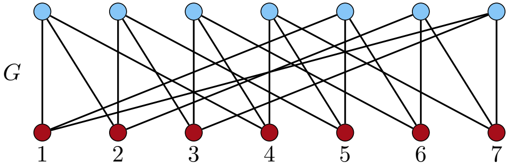
Lemma (2) [Existence of $(3,2)$-SS]

- For infinitely many k's, there is a $(3,2)$-SS $\mathcal{T}$ of size $\Omega(k^2)$.

Construction of G & $(2-\varepsilon, \Omega(k^2))$-DAM proof

- By Lemma (2), construct a $(3,2)$-SS $\mathcal{T}$ of size $\Omega(k^2)$.
- For each 3-subset of $\mathcal{T}$, build a star with terminals as leaves.
- Embed the stars in a bipartite graph as depicted below.



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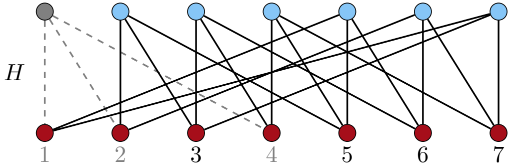

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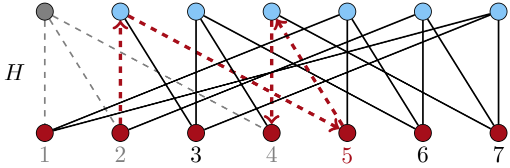
- Consider a minor H of G with strictly less non-terminals than G.
- By Lemma (1) and the structure of H, distortion must be ≥ 2.



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Open Questions

- For various classes of graphs, we showed lower/upper bounds on the number of non-terminal needed for achieving small distortion.

Remaining interesting questions:

- Can we improve the trade-off between distortion and size in the current lower and upper bounds?
  - Recall that for distortion $3-\varepsilon$, the LB is $\Omega(k^{6/5})$ while for distortion 3 the UB is $O(k^3)$.
- Do general graphs admit a $(\mathcal{O}(1), \widetilde{\mathcal{O}}(k^2))$-DAM?
- Can we prove a super-constant lowerbound for the $(\alpha, 0)$-DAM (SPR problem)?
  - The best LB is 8 while the best UB is $\mathcal{O}(\log^5 k)$.

Thank You! Questions?