# Proof Complexity Modulo the Polynomial Hierarchy: Understanding Alternation as a Source of Hardness

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# SAT and QBF

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Note: SAT treated as a black-box oracle by QBF solvers
(e.g. QBF solver *sKizzo* - Benedetti '05)

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- Connection to separation of complexity classes

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This also clashes with the QBF view of SAT as an oracle.

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- Result: straightforward to define ensembles that have poly bd proofs on any set of QBFs with bounded alternation

  So, proof size lower bounds address the ability to handle alternation

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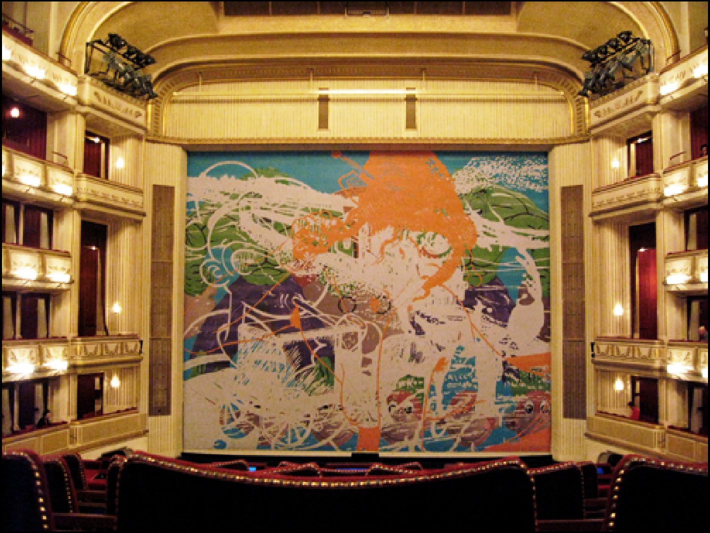
This talk: focus on 1 and 2.



Act: Framework

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Def (simplified): A proof system ensemble for a language $L$ is a sequence $(L_k)_{k \geqslant 1}$ of langs in PH such that:

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Def: Let $Z$ be a set of functions $\mathbb{N} \to \mathbb{N}$. (eg: $Z = \Omega(2^n)$)
A pf system ensemble $(L_k)_{k \geqslant 1}$ requires proofs of size $Z$ on instances $\Phi_1, \Phi_2, \ldots$ if $\forall k \geqslant 1, \exists z \in Z$ where

$$(\forall n \geqslant 1, \forall \pi) \qquad (\Phi_n, \pi) \in L_k \Rightarrow |\pi| \geqslant z(n)$$

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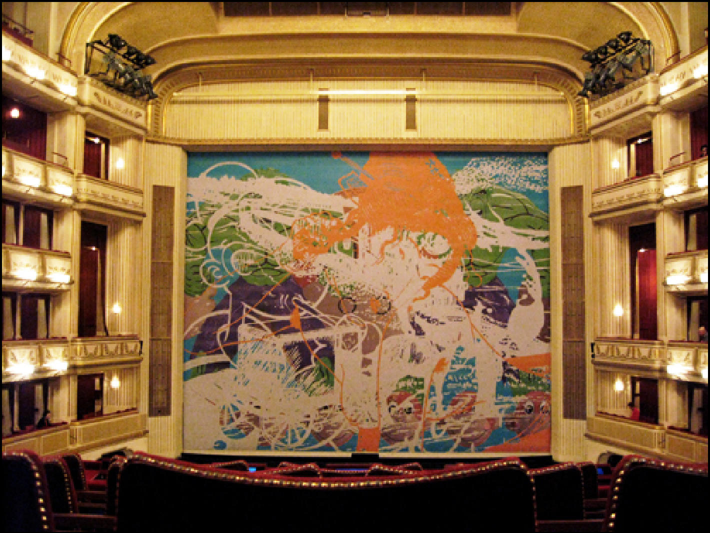
Note: $\exists$ poly bd ensemble for $\overline{\text{QBF}} \Leftrightarrow$ PSPACE $\subseteq$ PH

Relationship between this framework & PH vs. PSPACE qtn

is analogous to

the relationship between SAT proof complexity &
NP vs. coNP qtn



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Each $H(\Phi, \Pi_k)$ will give us a pf system

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To use prop: need to detect when $\Phi[a]$ is false
...but this is hard in general!

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Example relaxations: $\forall y \forall y' \exists x_1 \exists x_2 \exists x_3 \psi$, $\exists x_1 \forall y' \exists x_2 \forall y \exists x_3 \psi$

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We have $H(\Phi, \Pi_2) \subseteq H(\Phi, \Pi_3) \subseteq \cdots$

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- This makes sense even if $\Phi$ is not clausal, i.e., even if $\Phi$ has the form $Q_1 v_1 \ldots Q_n v_n$(circuit)
- This way of “lifting” to an enhanced set of clauses can be used to define relaxed versions of any clause-based QBF proof system

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2. Definition of *relaxing QU-resolution*,
   a particular ensemble obtained by “lifting” QU-resolution
3. Two technical results on relaxing QU-resolution:
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   exponential separation of general/tree-like versions

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What constitutes a good/reasonable/natural/etc. definition of a proof system ensemble?

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***end [fin]***