On the Size and the Approximability of Minimum Temporally Connected Subgraphs

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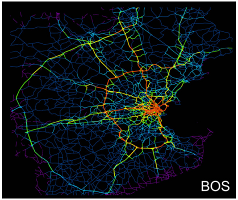
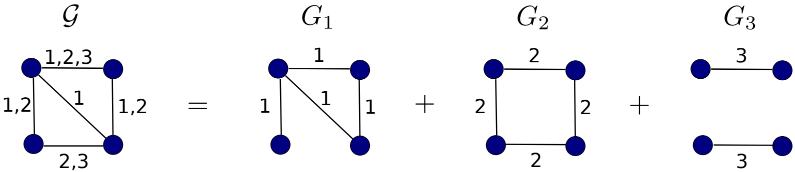
Motivation

- Graphs are a very useful abstract model...
- ...but traditional connectivity is a static measure
- In real-world applications connectivity "changes" through time
  - Transportation networks: Roads temporarily closed for maintenance
  - Social networks: Social interactions are not invariant
  - More generally, distributed computation and information spreading



Temporal graphs

- We need to generalize our model to graphs that capture changes in time...
- ...**Temporal** graphs!
- A sequence of usual undirected weighted graphs with invariant vertex set: $(G_t(V, E_t, w_t))_{t \in [L]}$ (L is called *lifetime*)

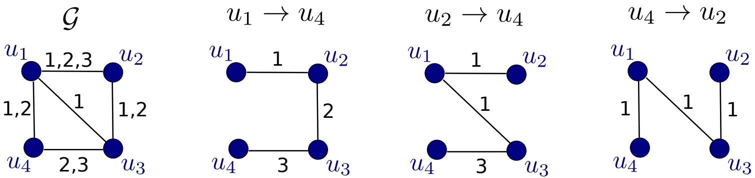


Temporal graphs (cont.)

- The *underlying* (usual) graph is $G(V, E)$ with $E = \bigcup_{t \in L} E_t$
- *Simple* temporal graph: Each edge available for at most one time moment
- Denote $N = |V|, M = \sum_{t \in [L]} |E_t|$

Temporal path

- *Temporal path* from $u_1 \in V$ to $u_k \in V$: A path from u_1 to u_k in G, but edges also equipped with non-decreasing time labels along the path



- Formally: $u_1, (e_1, t_1), u_2, (e_2, t_2), \ldots u_k$, where $t_i \leq t_{i+1}$ and $\{u_i, u_{i+1}\} \equiv e_i \in E_{t_i}$
- Intuitively: Starting at u_1 at time 0, we eventually reach u_k

Temporal connectivity

- To study connectivity, we need temporal analogues of connectivity and minimum connectivity certificates (minimum spanning trees)
- **Temporal connectivity**: There is a temporal path between each (ordered) pair of vertices
- **Minimum Temporally Connected Subgraph** (Spanning tree analogue): A temporal graph $\mathcal{G}'(V, E', w)$ with $E'_t \subseteq E_t, \forall t \in [L]$ such that $\mathcal{G}'$ is temporally connected, where $\sum_{t \in L} w_t(E'_t)$ is minimum
- Intuitively: We "buy" each edge for a subset of time instants, and still preserve temporal connectivity

Previous work

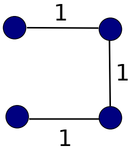
- [Berman, '96] presented an algorithm for reachability by temporal paths and proved a temporal analogue of the max-flow min-cut theorem
- [Kempe, Kleinberg, Kumar, '00] investigated the temporal generalization of certain graph properties (e.g Menger's theorem not true), showed worst-case MTCS size is $\Omega(N \log N)$ and $O(N^2)$
- [Mertzios, Michail, Chatzigiannakis, Spirakis, '13] proved that a variant of Menger's theorem holds

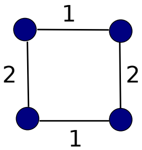
Previous work (cont.)

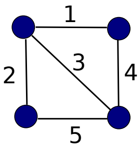
- [Erlebach, Hoffmann, Kammer, '15] showed that shortest *exploration schedules* can require $\Theta(N^2)$ steps
- [Akrida, Gasieniec, Mertzios, Spirakis, '15] showed a $\Theta(N \log N)$ lower bound on MTCS size by *strict* temporal paths, on graphs with distinct time labels. Also showed that computing the max number of edges to remove to preserve connectivity is APX-hard

Minimum Temporally Connected Subgraph size

- In usual graphs, all minimum connectivity certificates have size $n-1$
- In temporal graphs, this is not the case!

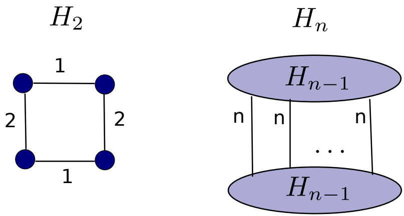






Minimum Temporally Connected Subgraph size (cont.)

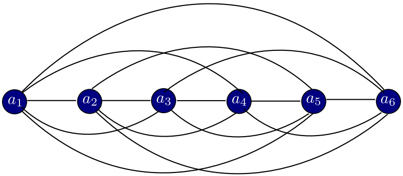
- Natural question (posed by Kempe et al (STOC 2000)): What is the worst-case size of a minimum temporally connected subgraph? (For *simple* temporal graphs)
- At least $\Omega(N \log N)$:



- We bridge the $\Omega(N \log N)$-$O(N^2)$ gap by showing that it is $\Theta(N^2)$

Lower bound on MTCS size

- We need a **dense**, simple, temporally connected graph, where the deletion of any edge breaks temporal connectivity
- First part: A linear number of "independent" paths, each of linear size, that share the same vertex set of size N.
- Partition a complete graph into $\frac{N}{2}$ Hamiltonian paths



Lower bound on MTCS size (cont.)

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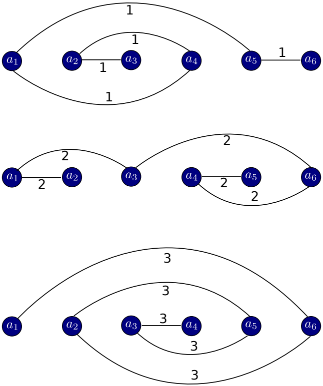
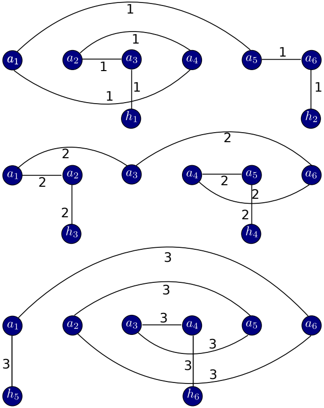


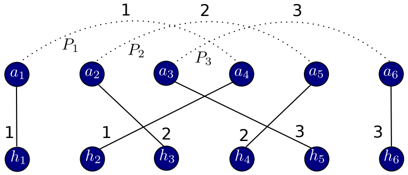
Lower bound on MTCS size (cont.)

- Connect N extra vertices, each to the endpoint of one path, with the same label as the path
- All $N + 1$ edges of the i-th Hamilton path have time label i



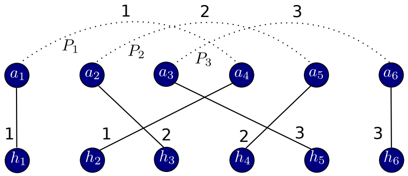
Lower bound on MTCS size (cont.)

- First (density) part: A linear number of "independent" paths, each of linear size, that share the same vertex set of size N.



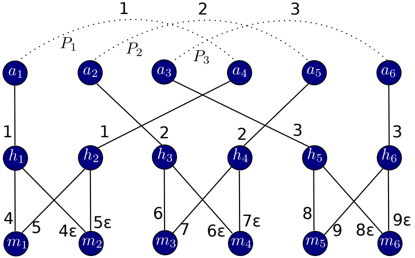
Lower bound on MTCS size (cont.)

- Second (interconnection) part: Connect the h_i's to each other
- Add N extra nodes, but careful not to introduce alternative temporal paths between h_{2i} and h_{2i-1} (technical)
- Each pair of these extra nodes correspond to a specific Hamilton path, are seen as an "entry" and an "exit" node respectively (to/from the path's endpoints)



Lower bound on MTCS size (cont.)

- Second (interconnection) part: Connect the h_i's to each other
- Add N extra nodes, but careful not to introduce temporal paths between internal vertices of Hamilton paths (technical)
- Each pair of these extra nodes correspond to a specific Hamilton path, are seen as an "entry" and an "exit" node respectively (to/from the path's endpoints)



Lower bound on MTCS size (cont.)

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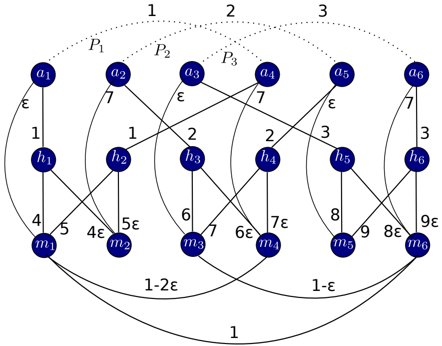
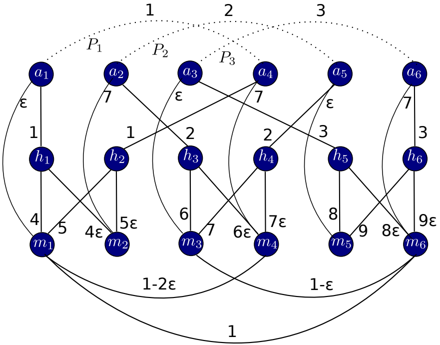


Lower bound on MTCS size (cont.)

- Remove any edge with label i from the first part $\Rightarrow$ impossible to move from h_{2i} to h_{2i-1}
- Construction forces each temporal path to start and end at time i
- Therefore all $\Theta(N^2)$ edges of the first part are essential



Lower bound on MTCS size (cont.)

- Summary: A simple, minimal, temporally connected graph with $\Theta(N)$ vertices and $\Theta(N^2)$ edges
- Note: We can make the size of the *MTCS* linear, by changing just one label in the above construction!

Reminder

- **Minimum Temporally Connected Subgraph** (Spanning tree analogue): A temporal graph $\mathcal{G}'(V, E', w)$ with $E'_t \subseteq E_t, \forall t \in [L]$ such that $\mathcal{G}'$ is temporally connected, where $\sum_{t \in L} w_t(E'_t)$ is minimum

$r - MTC$ and MTC

- What's next? Study the problem of computing Minimum Temporally Connected Subgraphs
- 2 useful variations:
  - $r - MTC$: Compute a minimum weight temporal subgraph, so that there are temporal paths from a fixed $r \in V$ to all $u \in V$
  - MTC: Compute a minimum weight temporally connected subgraph
- Both problems are very hard

$r - MTC$ and MTC (cont.)

- Central idea of all reductions: Time dependence induces implicit direction
- Temporal connectivity problems are closely related to directed *Steiner* problems

$r - MTC$ approximability

$r - MTC$:

- Optimal solution is a tree
- Essentially equivalent to Directed Steiner Tree
- No $O(\log^{2-\epsilon} N)$ approximation unless $NP \subseteq ZTIME(n^{poly \log n})$ (reduction from DST)
- $O(N^{\epsilon})$ approximation $\forall \epsilon$ (reduction to DST)
- Easy to solve in polynomial time for unweighted graphs (Prim-like algorithm)

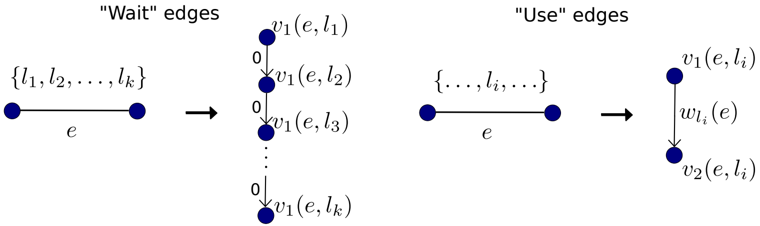
MTC approximability

MTC:

- No $O(2^{\log^{1-\epsilon} N})$ approximation unless $NP \subseteq DTIME(n^{poly \log n})$ (reduction from Label Cover)
- Trivial $O(N^{1+\epsilon})$ approximation: Run $r - MTC$ algorithm for each node as source
- APX-hard even for unweighted graphs (reduction from $\{1,2\}$-Steiner tree), but trivial L-approximation (pick any spanning tree at each instant)

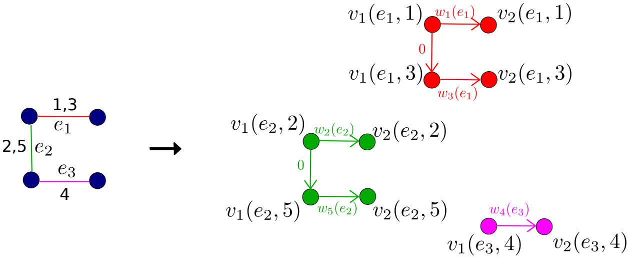
$MTC \rightarrow DSF$ reduction

- All *MTC* hardness results hold even for constant lifetime
- Better approximation for *MTC* for sparse temporal graphs with a reduction to Directed Steiner Forest
- Create two nodes for each temporal edge (edge-label pair)
- Add "wait" edges and "use" edges



$MTC \rightarrow DSF$ reduction (cont.)

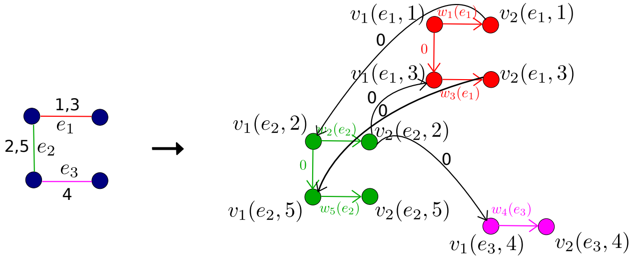
Example



- Add "move" edges (move from one edge to another)
- For near-linear number of temporal edges and polylogarithmic max degree, we get an $O(N^{2/3+\epsilon})$ approximation
- An $O(N^{1/6})$ approximation for the general MTC would improve the best-known $\widetilde{O}(N^{1/3})$ approximation for MIN-REP Label Cover

$MTC \rightarrow DSF$ reduction (cont.)

Example



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$r - MTC$, MTC in restricted graph families

- We can do much better if we restrict the underlying graph family
- $r - MTC$ in polynomial time for bounded treewidth underlying graphs
- MTC in polynomial time for temporal trees
- MTC 2-approximable for temporal cycles

Future work

- General *MTC* is too hard, can we get a better constant than L for the approximability in the unweighted case?
- Or maybe some algorithm to recognize instances with quadratic size connectivity certificate
- Other special cases with good approximation ratio (Moreover, is the 2-approximation for the temporal cycle tight?)

Thank you!

Questions?