# The Complexity of Rational Synthesis

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ICALP, 12-15 July 2016

# Motivation

- Classical reactive system synthesis:
  - One system and one antagonist environment
  - Synthesize a system to ensure the specification

Environment Model $\parallel$ ? $\models$ Specification

  - Synthesis ≈ two-player zero-sum game

# Motivation

- Classical reactive system synthesis:
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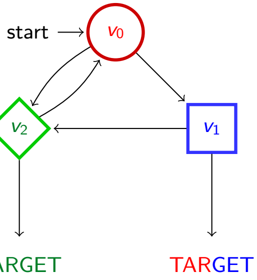
  Environment Model $\| \; ? \models$ Specification

  - Synthesis $\approx$ two-player zero-sum game

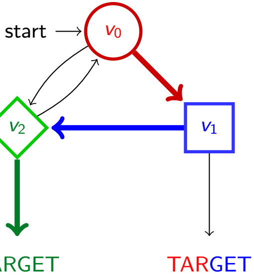
- Rational synthesis:
  - Multi-component environment
  - Non-antagonist objectives
  - Rational synthesis $\approx$ multiplayer turn-based game

# Multiplayer Games

- $\mathcal{G} = \langle \mathbb{P}, V, (V_i)_{i\in\Omega}, E, v_0, (\mathcal{O}_i)_{i\in\mathbb{P}} \rangle$ where $\mathbb{P} = \{0, 1, \ldots, k\}$ and $\mathcal{O}_i \subseteq V^\omega$



- Strategy of Player $i$ : $\sigma_i : V^* V_i \to V$
- Strategy profile $\bar{\sigma} = (\sigma_i)_{i \in \mathbb{P}}$,
- $pay(\bar{\sigma}) \in \{0,1\}^n$ s.t. $pay_i(\bar{\sigma}) = 1$ iff $out(\bar{\sigma}) \in \mathcal{O}_i$



# Strategies and Nash Equilibria

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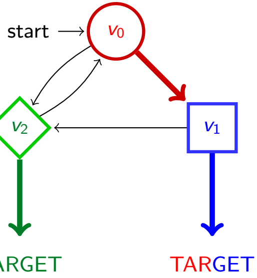

# Nash Equilibrium

**Definition (Nash Equilibrium (Nash51) )**

$\bar{\sigma}$ is Nash Equilibrium iff no incentive to deviate

$$pay_i(\bar{\sigma}_{-i}, \tau_i) \leq pay_i(\bar{\sigma}) \ \forall i \in \mathbb{P} \text{ and } \tau_i \text{ strategy of Player } i$$

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- $\bar{\sigma}$ is 0-fixed Nash Equilibrium iff

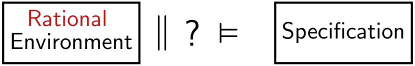
$$pay_i(\bar{\sigma}_{-i}, \tau_i) \leq pay_i(\bar{\sigma}) \ \forall i \in \mathbb{P} \setminus \{0\} \text{ and } \tau_i \text{ strategy of Player } i$$

# Rational Synthesis

Rational Environment ‖ ? ⊨ Specification

- Rationality = behavior according to a Nash equilibrium.

[1] D. Fisman, O. Kupferman, and Y. Lustig. Rational synthesis. CoRR, abs/0907.3019, 2009.
[2] O. Kupferman, G. Perelli, and M. Y. Vardi. Synthesis with rational environments. EUMAS 2014

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Definition (Rational Synthesis Problems)

**cooperative(CRSP):**[1] $\exists \sigma_0 \;\; \exists \bar{\sigma}$ s.t. $\langle \sigma_0, \bar{\sigma} \rangle \in NE_{\sigma_0} \;\wedge\; pay_0(\sigma_0, \bar{\sigma}) = 1$?

**non-cooperative(NCRSP):**[2] $\exists \sigma_0 \;\; \forall \bar{\sigma} \; : \; \langle \sigma_0, \bar{\sigma} \rangle \in NE_{\sigma_0} \rightarrow pay_0(\sigma_0, \bar{\sigma}) = 1$ ?

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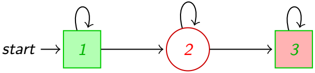
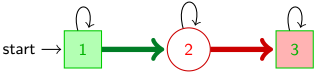
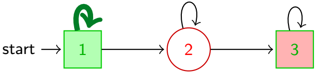
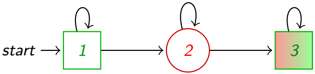
# Rational Synthesis - Example1




*Reachability objectives:* $R_{\bigcirc} = \{3\}$, $R_{\square} = \{1\}$

| Cooperative | non-Cooperative |
|---|---|
|  |  no solution |

# Rational Synthesis - Example2

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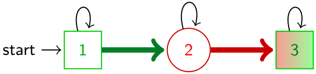
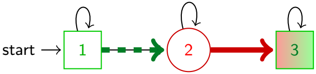
| **Cooperative** | **non-Cooperative** |
| --- | --- |
|   |   |

# Related work and goal

(LTL winning objectives)

- *Introduced by Kupferman et al. for LTL objectives*
- *Use Strategy Logic (SL) to characterize 0-fixed equilibria*
- *Reduce to Model Checking problem of SL formulas*
- 2EXPTIME-*complete for both cooperative and non-cooperative settings*

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(Goal)

- *Finely understand the computational complexities*
- *Study particular objectives (e.g. Safety, Reachability, Büchi,...)*

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# Cooperative Rational Synthesis

**cooperative:**[2] $\exists\sigma_0 \;\; \exists\bar{\sigma} \;\; s.t. \; \langle\sigma_0,\bar{\sigma}\rangle \in NE_{\sigma_0} \;\wedge\; pay_0(\bar{\sigma}) = 1?$

(Ummels[3])

$$For \; \vec{x},\vec{y} \in \{0,1\}^{k+1}, \; \exists \; a \; NE \; \bar{\sigma} \; s.t. \; \vec{x} \leq pay(\bar{\sigma}) \leq \vec{y}?$$

- Rational synthesis: $\vec{x} = (1,0,...,0)$, $\vec{y} = (1,1,...,1)$
- Nash equilibrium characterization on path properties
- Extend for Safety and Reachability

[2] O. Kupferman, G. Perelli, and M. Y. Vardi. Synthesis with rational environments. EUMAS 2014

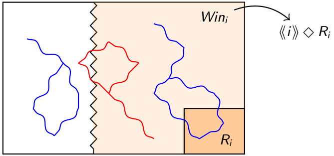
[3] M. Ummels. The complexity of Nash Equilibria in infinite multiplayer games. FOSSACS 2008

# LTL Characterization of 0-fixed Nash Equilibria

## For Safety, Reachability and tail objectives

- $Win_i$ = winning region for Player $i$ against all other players (2-player 0-sum game)
- If $\mathcal{O}_i = Reach(R_i)$ for some $R_i \subseteq V$ (similar for tail objectives) :

$$\bigwedge_{i=1}^{k}(\Box\neg R_i \rightarrow \Box\neg Win_i)$$



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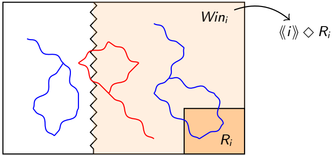
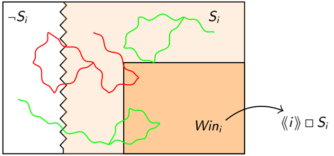



- If $\mathcal{O}_i = \text{Safe}(S_i)$ for some $S_i \subseteq V$:

$$\bigwedge_{i=1}^{k}((\neg Win_i \,\mathcal{U}\, \neg S_i) \vee \Box S_i)$$



# Cooperative Rational Synthesis

## For Safety, Reachability and tail objectives

Find $\pi \in V^\omega$ s.t.

- If $\mathcal{O}_i = Reach(R_i)$ for some $R_i \subseteq V$ (for tail objectives is similar) :

$$\pi \models \Diamond R_0 \wedge \bigwedge_{i=1}^{k} (\Box \neg R_i \rightarrow \Box \neg Win_i)$$

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*If it exists such $\pi$, it exists $\pi = x(y)^{\omega}$ with $|xy|$ polynomial in $\mathcal{G}$*

# Cooperative Rational Synthesis

## For Safety, Reachability and tail objectives

| | Cooperative | |
|---|---|---|
| | Unfixed | Fixed # Players |
| Safety | NP-c | PTIME-c |
| Reachability | NP-c | PTIME-c |
| Büchi | PTIME-c[3] | PTIME-c[3] |
| co-Büchi | NP-c[3] | PTIME-c |
| Parity | NP-c[3] | $UP \cap co - UP$, parity-h |
| Streett | NP-c[3] | NP[3], NP-hard |
| Rabin | $P^{NP}$, NP-h, coNP-h | $P^{NP}$, coNP-h |
| Muller | PSPACE-c | PSPACE-c |
| LTL | 2EXPTIME-C[2] | 2EXPTIME-C[2] |

- Compute winning sets for each player
- Test the existence of a lasso path satisfying the LTL characterization

# Non-Cooperative Rational Synthesis Problem

**non-cooperative:** $\exists \sigma_0 \quad \forall \bar{\sigma} \quad : \quad \langle \sigma_0, \bar{\sigma} \rangle \in NE_{\sigma_0} \rightarrow pay_0(\bar{\sigma}) = 1$ ?

- First attempt: two player zero-sum game with objective

$$\pi \models \Big( \bigwedge_{i=1}^{k} (\Box \neg R_i \rightarrow \Box \neg Win_i) \Big) \rightarrow \Diamond R_0$$

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Incorrect!

Fix $\sigma_0$. Only 0-fixed NE w.r.t. $\sigma_0$ should be considered !!!

# Automata-based solution for NCRSP

- Encode strategies of Player 0 as trees
- Define a nondeterministic tree automata to accept solutions

$$\mathcal{L}(\mathcal{T}) = \{t_{\sigma_0} \mid \sigma_0 \text{ is solution to NCRSP}\}$$

  - for each branch $\pi$ of $t_{\sigma_0}$ compatible to $\sigma_0$, check that:
    - $\pi \in \mathcal{O}_0$ or
    - $\pi$ not the outcome of a 0-fixed NE

      $\updownarrow$

      guess at least one player that wants to deviate from $\pi$
      and check he has a winning strategy under $\sigma_0$

# Automata-based solution for NCRSP

- States: $q = (W, D, v) \in 2^{\mathbb{P}} \times 2^{\mathbb{P}} \times V$
  - $W$ : the set of players that have winning strategy from $v$
  - $D$ : the set of players that have a winning deviation from the current prefix

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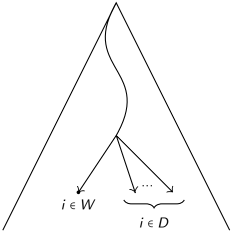
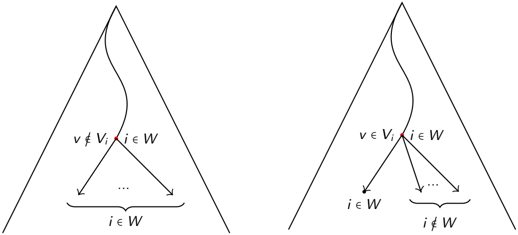
Update deviation information (D)




Update W information



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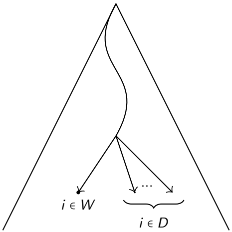


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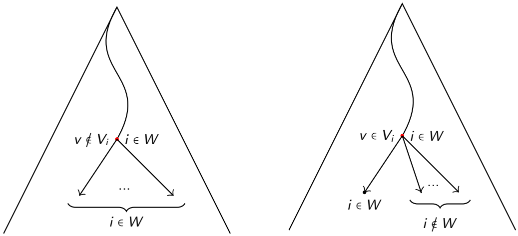


- Acceptance condition:
  - choice of W according to winning strategies
  - the accepted tree encodes a solution for NCRSP

# Non-Cooperative Rational Synthesis

| | Non-Cooperative | |
|---|---|---|
| | Unfixed | Fixed # Players |
| Safety | PSPACE-c | PTIME-c |
| Reachability | PSPACE-c | PTIME-c |
| Büchi | PSPACE-c | PTIME-c |
| co-Büchi | PSPACE-c | PTIME-c |
| Parity | EXPTIME, PSPACE-h | PSPACE, NP-h, coNP-h |
| Streett | EXPTIME,PSPACE-h | PSPACE-c |
| Rabin | EXPTIME, PSPACE-h | PSPACE-c |
| Muller | EXPTIME, PSPACE-h | PSPACE-c |
| LTL | 2EXPTIME-C[2] | 2EXPTIME-C[2] |

- on-the-fly emptiness checking
- careful analysis :
  - exponential size in $|\mathbb{P}|$
  - monotonicity properties along paths of accepting runs
- for fixed # of Players : polynomial size automaton

# Conclusions and Future Work

| | Cooperative | | Non-Cooperative | |
|---|---|---|---|---|
| | Unfixed # Players | Fixed # Players | Unfixed # Players | Fixed # Players |
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| co-Büchi | NP-c[3] | PTIME-c | PSPACE-c | PTIME-c |
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- Future work:
  - Imperfect information
  - Other notions of rationality

Thank you!