Piecewise Testable Separability for Regular Tree Languages

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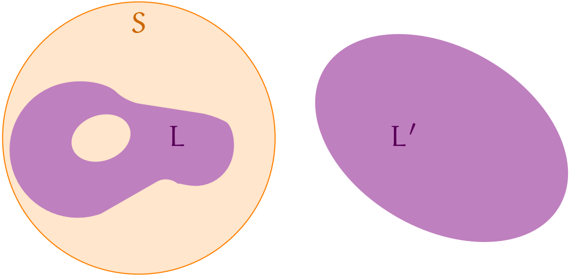
Separability

Consider two classes $\mathcal{C}$ and $\mathcal{S}$ of languages:

Problem ($\mathcal{S}$ Separability over $\mathcal{C}$)

input L *and* L' *from* $\mathcal{C}$

question $\exists S \in \mathcal{S} \,.\, L \subseteq S$ *and* $S \cap L' = \emptyset$?



Separability

Problem (PTL Separability over $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$)

input L *and* L' *from* $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$

question $\exists S \in \mathrm{PTL} \,.\, \mathrm{L} \subseteq S$ *and* $S \cap \mathrm{L}' = \emptyset$?

In this talk, $\mathcal{C} = \mathrm{Reg}(\mathrm{T}(\mathcal{F}))$: regular languages of finite ranked trees over a finite alphabet $\mathcal{F}$:

Theorem

PTL separability over $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$ is decidable.

Corollary (c.f. Place (2008))

PTL definability over $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$ is decidable.

Separability

Problem (PTL Definability over $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$)

input L *from* $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$
question $\mathrm{L} \in \mathrm{PTL}$?

In this talk, $\mathcal{C} = \mathrm{Reg}(\mathrm{T}(\mathcal{F}))$: regular languages of finite ranked trees over a finite alphabet $\mathcal{F}$:

Theorem
PTL separability over $\mathrm{Reg}(\mathrm{T}(\mathcal{F}))$ *is decidable.*

Corollary (c.f. Place (2008))
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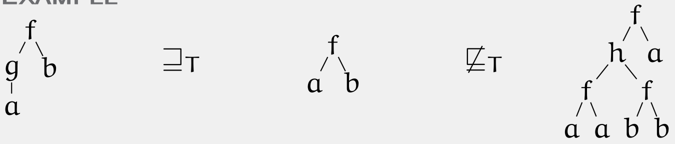
Homeomorphic Tree Embedding

$t \sqsubseteq_T t'$ iff $t = f(t_1, \ldots, t_r)$, $t' = g(t'_1, \ldots, t'_s)$, and

- $\exists 1 \leqslant i \leqslant s \,.\, t \sqsubseteq_T t'_i$, or
- $f = g$ (and thus $r = s$) and $\forall 1 \leqslant i \leqslant r \,.\, t_i \sqsubseteq_T t'_i$

Example



Piecewise Testable Languages

- n-piecewise equivalence: $t_1 \equiv_n t_2$ iff $\forall t$ of size $\leqslant n$, $t \sqsubseteq_T t_1$ iff $t \sqsubseteq_T t_2$
- $L \subseteq T(\mathcal{F})$ is a PTL iff $\exists n \,.\, L$ is a finite union of n-piecewise equivalence classes
- principal filter: $\uparrow t \stackrel{\text{def}}{=} \{t' \in T(\mathcal{F}) \mid t \sqsubseteq_T t'\}$

Fact

L is a PTL iff it is a finite Boolean combination of principal filters.

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L *is a PTL iff it is a finite Boolean combination of principal filters.*

Example

Consider $L \stackrel{\text{def}}{=} \{ f(g(a), b) \}$ over alphabet $\mathcal{F} \stackrel{\text{def}}{=} \{f^{(2)}, g^{(1)}, a^{(0)}, b^{(0)}\}$

$$L = \uparrow f(a, b) \quad \cap \quad \uparrow g(a)$$

$$\cap \quad T(\mathcal{F}) \setminus \uparrow g(b) \quad \cap \quad T(\mathcal{F}) \setminus \uparrow g(g(a))$$

$$\cap \quad T(\mathcal{F}) \setminus \uparrow f(a, f(a, b)) \quad \cap \quad T(\mathcal{F}) \setminus \uparrow f(f(a, b), a)$$

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Logical Characterisation

Consider the logic $\mathrm{FO}(<_{\mathrm{dfs}}, \sqcap, (P_f)_{f\in\mathcal{F}})$ quantifying over positions inside trees of $T(\mathcal{F})$:

- $x <_{\mathrm{dfs}} y$ iff position x appears before position y in a depth-first-search traversal of a tree
- $y \sqcap z = x$ iff x is the least common ancestor of y and z
- $P_f(x)$ iff x is labelled by f

Fact

$\sqsubseteq_T$ is the induced substructure ordering over $T(\mathcal{F})$ with signature $(<_{\mathrm{dfs}}, \sqcap, (P_f)_{f\in\mathcal{F}})$.

Fact

$L \subseteq T(\mathcal{F})$ is a PTL iff it is definable in $\mathcal{B}\Sigma_1(<_{\mathrm{dfs}}, \sqcap, (P_f)_{f\in\mathcal{F}})$.

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Example

$L \stackrel{\text{def}}{=} \left\{ \begin{array}{ccc} & f & \\ & / \quad \backslash & \\ g & & b \\ | & & \\ a & & \end{array} \right\}$ over $\mathcal{F} \stackrel{\text{def}}{=} \{f^{(2)}, g^{(1)}, a^{(0)}, b^{(0)}\}$ is defined by

$$
\begin{array}{lllll}
 & \exists x,y,z\,. & y \sqcap z = x \wedge y <_{\text{dfs}} z & \wedge & P_f(x) \wedge P_a(y) \wedge P_b(z) \\
\wedge & \exists x,y\,. & x \sqcap y = x & \wedge & P_g(x) \wedge P_a(y) \\
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$$L \stackrel{\text{def}}{=} \left\{ \begin{array}{c} f \\ g \quad b \\ a \end{array} \right\} \text{ over } \mathcal{F} \stackrel{\text{def}}{=} \{f^{(2)}, g^{(1)}, a^{(0)}, b^{(0)}\} \text{ is defined by}$$

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Related Work

over $\mathrm{Reg}(\Sigma^*)$	over $\mathrm{Reg}(T(\mathcal{F}))$
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PTL definability (Simon, 1975)	
Σ_2 definability (Arfi, 1987)	
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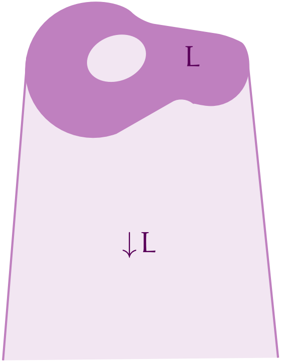
Order-Theoretic Viewpoint

downward closure $\downarrow L \stackrel{\text{def}}{=} \{t \in T(\mathcal{F}) \mid \exists t' \in L \,.\, t \leqslant t'\}$



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Theorem (Higman, 1952)
$(T(\mathcal{F}), \sqsubseteq_T)$ *is a well-quasi-order (wqo).*

finite bad sequences any sequence $t_0, t_1, t_2, \ldots$ with $\forall i < j, t_i \not\sqsubseteq_T t_j$, is finite

descending chain property all the descending chains $D_0 \supsetneq D_1 \supsetneq D_2 \supsetneq \cdots$ of downwards-closed subsets are finite

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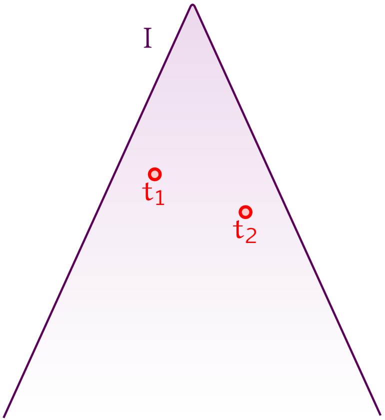
Ideal Decompositions

(Bonnet, 1975; Finkel and Goubault-Larrecq, 2009)

- Directed set Δ
  non-empty and $\forall t_1, t_2 \in I$, $\exists t. t_1 \leqslant t$ and $t_2 \leqslant t$
- Ideal I
  downwards-closed and directed
- Examples
  - $\downarrow t \in \mathrm{Idl}(T(\mathcal{F}))$ for any t in $T(\mathcal{F})$
  - $(g(\square))^* \cdot a \in \mathrm{Idl}(T(\mathcal{F}))$ for $g \in \mathcal{F}_1$ and $a \in \mathcal{F}_0$
- Canonical Decompositions
  if $D \subseteq T(\mathcal{F})$ is downwards-closed, then $D = I_1 \cup \cdots \cup I_n$



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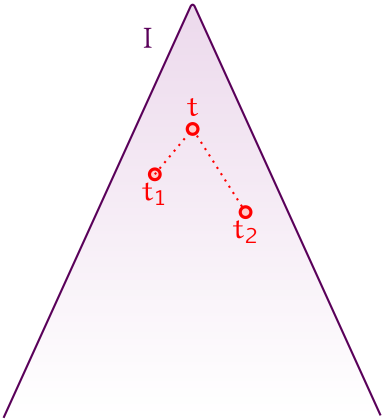

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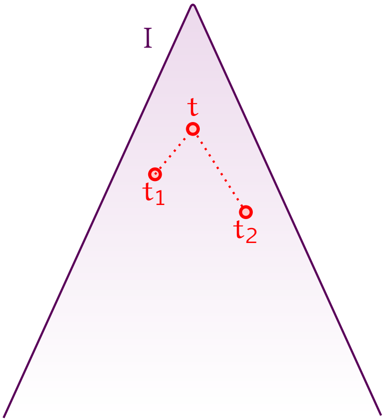

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- Directed set Δ
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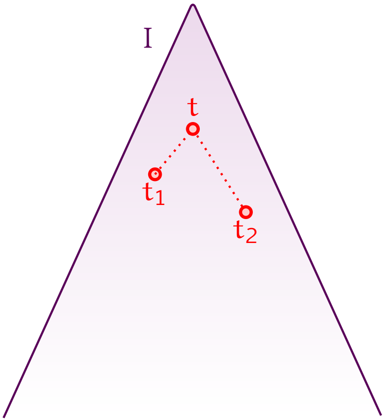

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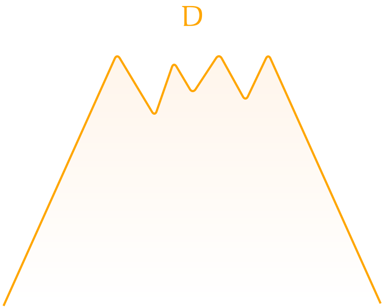

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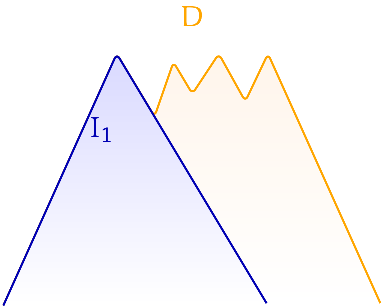

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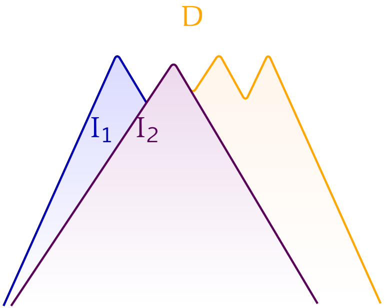

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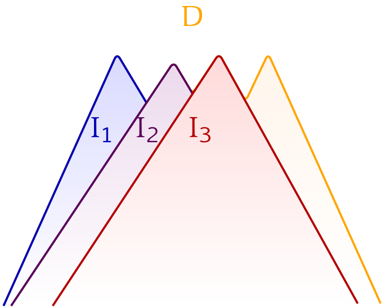

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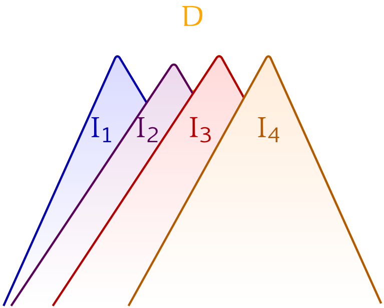

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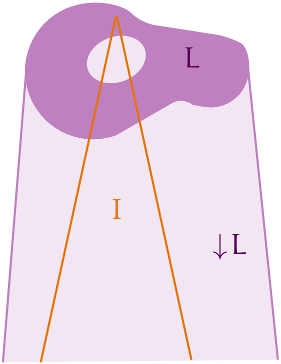

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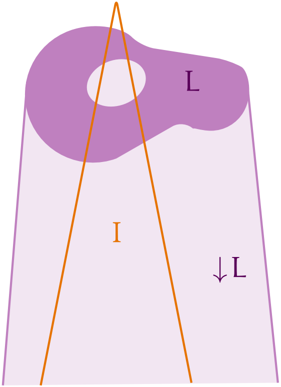

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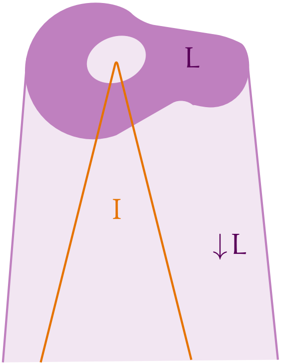

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Key Lemma
L and L' are not PTL separable over $T(\mathcal{F})$ iff $\mathrm{Adh}(L) \cap \mathrm{Adh}(L') \neq \emptyset$.

- over any wqo
- simple proof

Reduction to Adherence Membership

RE enumerate PTLs S until $L \subseteq S$ and $S \cap L' = \emptyset$

coRE enumerate ideals I until $I \in \mathrm{Adh}(L)$ and $I \in \mathrm{Adh}(L')$

Effective Ideal Representations

Theorem
The ideals of $(T(\mathcal{F}), \sqsubseteq_T)$ *can be represented as particular tree regular expressions.*

Hence:

- ideals I can be enumerated
- whether $I \in \mathrm{Adh}(L)$, i.e. whether $I \subseteq \downarrow(I \cap L)$, is decidable for $L \in \mathrm{Reg}(T(\mathcal{F}))$

Bonus Result

Generalising Czerwiński, Martens, van Rooijen, Zeitoun, and Zetzsche (2015):

Bonus Theorem

Under some 'mild assumptions,' the following decision problems are Turing-equivalent over a wqo:

- *given* I *and* L*, deciding whether* $I \in \mathrm{Adh}(L)$
- *given* L*, computing the ideal decomposition of* $\downarrow L$

Open Questions

- unranked trees (Bojańczyk, Segoufin, and Straubing, 2012, for definability)
  - preserves wqo for the 3 'good' signatures
  - ideal representations? (see Finkel and Goubault-Larrecq, 2009)
- other relational signatures over $T(\mathcal{F})$
  - Bojańczyk et al. (2012) consider 4 signatures, one of which is not wqo
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Final Remarks

- order-theoretic framework for piecewise separability
- ideals and downwards-closed sets as algorithmic tools
  - here, piecewise-testable separability
  - forward analysis (Finkel and Goubault-Larrecq, 2009, 2012)
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Proof of the Key Lemma (1/3)

- combinatorial qo $(X, \leqslant)$:

$$\forall n \,.\, X_{\leqslant n} \stackrel{\text{def}}{=} \{x \in X \mid \|x\| \leqslant n\} \text{ is finite}$$

- n-equivalence over $(X, \leqslant)$:

$$x \equiv_n y \quad \text{iff} \quad \forall z \in X_{\leqslant n} \,.\, z \leqslant x \text{ iff } z \leqslant y$$

Lemma (Folklore)

L and L' are not PTL separable over a qo $(X, \leqslant)$ if and only if there exist two sequences of elements $(x_n)_{n\in\mathbb{N}}$ in L and $(x'_n)_{n\in\mathbb{N}}$ in L' s.t. $\forall n \,.\, x_n \equiv_n x'_n$.

Proof of the Key Lemma (2/3)

Key Lemma

L and L' are not PTL separable over a wqo $(X, \leqslant)$ iff $\exists \Delta \subseteq L, \Delta' \subseteq L'$ directed s.t. $\downarrow\Delta = \downarrow\Delta'$.

Proof of 'if'.

Let $\Delta \subseteq L$ and $\Delta' \subseteq L'$ be directed s.t. $\downarrow\Delta = \downarrow\Delta'$. Let us show that $\forall n \in \mathbb{N}. \exists x_n \in L, x'_n \in L'. x_n \equiv_n x'_n$

- let $I \stackrel{\text{def}}{=} \downarrow\Delta = \downarrow\Delta'$
- $\forall n \in \mathbb{N}. I \cap X_{\leqslant n}$ is finite
- $\forall z \in I \cap X_{\leqslant n}$,
  - $\exists x_z \in \Delta. z \leqslant x_z$
  - $\exists x'_z \in \Delta'. z \leqslant x'_z$
- since Δ and Δ' are directed, $\exists x_n \in \Delta$ and $\exists x'_n \in \Delta'$ greater or equal to all those finitely many x_z and x'_z when z ranges over $I \cap X_{\leqslant n}$
- Then $x_n \equiv_n x'_n$: $\forall z \in X_{\leqslant n}$,
  - either $z \in I$ and then both $z \leqslant x_z \leqslant x_n$ and $z \leqslant x'_z \leqslant x'_n$,
  - or $z \notin I$ and then both $z \not\leqslant x_n$ and $z \not\leqslant x'_n$ since I is downwards-closed □

Proof of the Key Lemma (3/3)

Key Lemma

L and L' are not PTL separable over a wqo $(X,\leqslant)$ iff $\exists\Delta\subseteq L,\Delta'\subseteq L'$ directed s.t. $\downarrow\Delta=\downarrow\Delta'$.

Proof of 'only if'.

Assume L and L' are not PTL separable, hence there exist two infinite sequences $(x_n)_n$ in L and $(x'_n)_n$ in L' with $x_n\equiv_n x'_n$ for every n.
Let us consider the infinite sequence of *pairs* $(x_n,x'_n)_{n\in\mathbb{N}}$.

- by Dickson's Lemma, $X\times X$ is a wqo for the product ordering, hence there exists an infinite sequence of indices $i_0<i_1<\cdots$ such that $x_{i_j}\leqslant x_{i_{j+1}}$ and $x'_{i_j}\leqslant x'_{i_{j+1}}$ for every $j\in\mathbb{N}$.
- define $\Delta\stackrel{\text{def}}{=}\{x_{i_j}\mid j\in\mathbb{N}\}$ and $\Delta'\stackrel{\text{def}}{=}\{x'_{i_j}\mid j\in\mathbb{N}\}$.
- Δ and Δ' are directed
- by symmetry, it remains to show that $\downarrow\Delta\subseteq\downarrow\Delta'$:
  - consider some $x_{i_j}\in\Delta$
  - then there exists some index $i_k>\max(i_j,\|x_{i_j}\|)$
  - hence $x_{i_j}\leqslant x_{i_k}$,
  - and since $x_{i_k}\equiv_{i_k}x'_{i_k}$, $x_{i_j}\leqslant x'_{i_k}$ and thus $x_{i_j}\in\downarrow\Delta'$ □

Assumptions of Bonus Theorem

$\mathcal{C}$ is PTL effective

- Given a PTL $\mathcal{S}$ and $L \in \mathcal{C}$, $\mathcal{S} \cap L \in \mathcal{C}$ is computable
- Given $L \in \mathcal{C}$, $L = \emptyset$ is decidable

$(X, \leqslant)$ is fully effective

XI compute the ideal decomposition of X;

IC decide whether $I \subseteq J$ given two ideals $I, J \in \mathrm{Idl}(X)$;

II compute the ideal decomposition of $I \cap J$ given two ideals $I, J \in \mathrm{Idl}(X)$;

CU′ compute the ideal decomposition of $X \setminus \uparrow x$ given an element $x \in X$.

Proof of Bonus Theorem

Bonus Theorem

Let $(X,\leqslant)$ be a fully effective wqo and $\mathcal{C} \subseteq 2^X$ be PTL-effective over $(X,\leqslant)$. Then the adherence membership problem and the ideal decomposition problem are Turing-equivalent.

From adherence membership to ideal decomposition.
Using CD, I has a representation as a PTL, hence $I \cap L$ is in $\mathcal{C}$ and can be constructed. An ideal decomposition $I_1 \cup \cdots \cup I_k$ of $\downarrow(I \cap L)$ is then computed using the oracle, and since I is irreducible the test $I \subseteq \downarrow(I \cap L)$ is handled by k calls to IC. □

From ideal decomposition to adherence membership.

- $D_0 \stackrel{\text{def}}{=} X$ using XI
- $\forall k = 0,1,\ldots$, if $\exists I$ in the decomposition of D_k s.t. $I \not\subseteq \downarrow L$
  - enumerate $x \in I$ using IM until $\uparrow x \cap L = \emptyset$, which is decidable since $\uparrow x$ is a PTL,
  - set $D_{k+1} \stackrel{\text{def}}{=} D_k \setminus \uparrow x$ using II and CU′.
- correctness: by (Leroux and S., 2015, Lemma IV.7) $I \not\subseteq \downarrow L$ iff $I \notin \mathrm{Adh}(L)$
- termination: descending chain property □

Expressions for Downwards-Closed Sets

Definition (STRE)

$$S ::= P_1 + \cdots + P_m \quad P ::= f^?(S_1, \cdots, S_r) \mid C^*.S$$
$$C ::= A + \cdots + A \quad A ::= f(S_{\square 1}, \cdots, S_{\square r}) \quad S_\square ::= S \mid \square$$

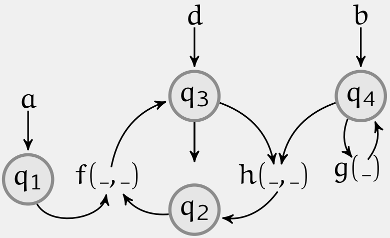
- $f \in \mathcal{F}_r$
- $+$ is associative and commutative with 0 denoting the empty sum
- $\square \notin \mathcal{F}$ is a *single* placeholder

Expressions for Downwards-Closed Sets

Proposition

Every STRE defines a downwards-closed language of $(T(\mathcal{F}), \sqsubseteq_T)$. Every downwards-closed language of $(T(\mathcal{F}), \sqsubseteq_T)$ is the language of some STRE.

Example



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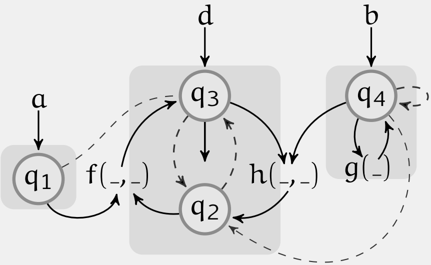


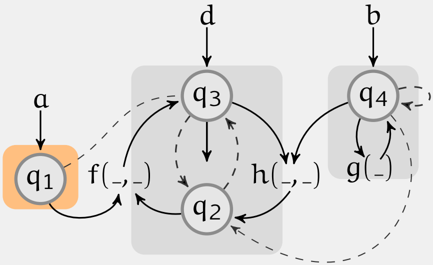
$S_1 = a^? \quad S_4 = (b + g(\square))^*.0 \quad L = (d + f(S_1, \square) + h(\square, S_4))^*.0$

Expressions for Downwards-Closed Sets

Proposition

Every STRE defines a downwards-closed language of $(T(\mathcal{F}), \sqsubseteq_T)$. Every downwards-closed language of $(T(\mathcal{F}), \sqsubseteq_T)$ is the language of some STRE.

Example



$S_1 = a^?$ $\quad S_4 = (b + g(\square))^*.0 \quad L = (d + f(S_1, \square) + h(\square, S_4))^*.0$

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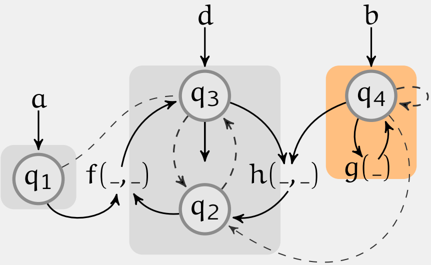

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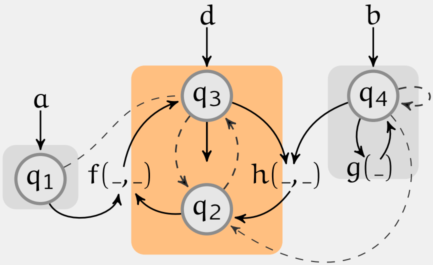

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Expressions for Ideals (1/2)

$A = f(S_{\square 1}, \cdots, S_{\square n})$ is

- $\square$-linear iff at most one $S_{\square i}$ is the placeholder $\square$
- $\square$-generated iff at least one $S_{\square_i}$ is $\square$
- empty iff some $S_{\square i} \neq \square$ has an empty language

Lifted to $C = A_1 + \cdots + A_m$ if every A_i has the property.

Expressions for Ideals (2/2)

Definition (Tree Products)

The tree products are the STREs of the form:

- $f^?(P_1, \cdots, P_n)$ where $P_1, \ldots, P_n$ are tree products, or
- $C^*.(P_1 + \cdots + P_n)$ where $n \in \mathbb{N}$, $P_1, \ldots, P_n$ are tree products, $C = \sum_{i=1}^{m} f_i(P_{\square i1}, \cdots, P_{\square i n_i})$, each pattern $P_{\square ij}$ is either a tree product or the placeholder $\square$, C is $\square$-generated, and one of the following conditions holds

1. C is not $\square$-linear and $n \geqslant 1$, or
2. C is not $\square$-linear, $n = 0$, and $P_{\square ij} \neq \square$ for some i, j, or
3. C is $\square$-linear and $n = 1$.

Theorem

The ideals of $T(\mathcal{F})$ are exactly the languages of tree products.